THE WEST UNIVERSITY OF TIMIȘOARA
DOCTORAL SCHOOL OF MATHEMATICS

PHD THESIS

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Contents

Abstract

The objective of the present thesis concerns Jacobi-type constructs that exhibit the main properties of standard Jacobi structures. This objective is fulfilled via three directions, as follows.

First, the Jacobi concept is scrutinized at the linear (algebraic) level by studying the category of Jacobi vector spaces. It is shown that, as in the smooth situation, it encompasses both locally conformal and contact linear structures. There are touched several aspects in this endeavor, like transitivity, Poissonization, coisotropy, product structures, transversals. The cornerstone of the first direction is represented by the linear version of contact dual pairs, which are deeply scrutinized.

Second, combining the twisted Jacobi pair [62] with that of a Poisson structure with a (closed) 3-form background [70], alias twisted Poisson, a new structure is proposed and investigated. This is called a Jacobi pair with background and consists of a bi-vector, and a vector field in the presence of a background made up of a 3-form and a 2-form. This new concept 'captures' twisted Jacobi pairs via exact backgrounds (through an appropriate homological derivation). It is shown that the manifolds equipped with such pairs enjoy the main properties of twisted Jacobi pairs: i) are in a one-to-one relation with the homogeneous Poisson structures with *arbitrary* 3-form background, and ii) display completely integrable characteristic distributions. This direction concludes with the analysis of twisted contact dual pairs.

Third, starting from the line bundle formulation of the Jacobi pair, the Jacobi bundle with background is introduced and studied. This is the natural global formulation of Jacobi pair with background concept. The structure 'lives' on (generally) non-trivial line bundles $L \to M$ and is equipped with a first-order bi-differential operator on the line bundle $J \in \mathcal{D}^2 L$ and an L-valued Atiyah 3-form $\Psi \in \Omega^3_L$ that are 'compatible' via a Maurer-Cartan-like consistency condition. Here, the exactness of twisted Jacobi bundles come from the exactness of the Atiyah 3-form $\Psi \in \Omega^3_L$ with respect to the homological degree 1-derivation in der-complex, d_L. It is shown that Jacobi bundles with background exhibit completely integrable characteristic distributions with characteristic leaves transitive Jacobi structures with background. The direction concludes with analyzing transitive Jacobi bundles with background establishing their connection with locally conformal symplectic structures with background and twisted contact structures.

Chapter 1

Introduction

Devised in the middle of the XIX-th century as the canonical first-order formulation for Lagrangian mechanics, Hamiltonian mechanics has been proved to be one of the most inspiring ideas for our days' fundamental science. In physics, this was the key ingredient of the quantization of systems admitting classical description [25, 26], and also, via Dirac algorithm, it put on stage the systems endowed with gauge symmetries. In mathematics, it has offered an 'odd' version to Riemannian geometry– the symplectic geometry, whose contravariant (possible degenerate) version– Poisson geometry sits, through a canonical structure, at the foundation of various quantization schemes [7, 35].

In the early '70s, after the advent of infinitesimal version of the groupoids [66], the idea of a Poisson-like bracket has been 'extended' to the sections in a vector bundle [42], giving rise to the local Lie algebras. Soon after that, Jacobi structures have emerged [46, 33]. Initially, the concept, known today as the Jacobi pair [23, 73], was expressed by a bi-vector and a vector field. Jacobi pairs enjoy two main properties: i) are in a one-to-one relation with the homogeneous Poisson structures with *arbitrary* 3-form background, and ii) display integrable characteristic distributions with characteristic leaves either cooriented locally conformal symplectic manifolds or cooriented contact ones. The deep understanding of Lie algebroids [53, 22], supplemented with a fine analysis [56] of the old concept of local Lie algebra [42], have currently led to the (globally unifying) line bundle perspective on Jacobi structures [23, 32, 56, 73]. Within this framework, the Jacobi pairs are nothing but Jacobi structures over trivial line bundles.

The issue of deforming Lie algebras, a problem that has been exploited in physics both at the construction of consistent interactions between gauge fields [3] and the quantization within the BV scheme [30], further provoked the birth of twisted-type structures like Poisson [70] and Jacobi [62]. It is noteworthy that the second kind of twisting, namely Jacobi, has been done [62] within the trivial line bundle, the reason for why such structures will be addressed as twisted Jacobi pairs in the sequel. Using the powerful machinery of the integrability of Dirac(-Jacobi) structures [21, 79], it has been shown that both twisted Poisson [70] and twisted Jacobi [62] display characteristic distributions that are completely (Stefan-Sussmann [71, 72]) integrable.

The present work plants a little mustard seed in this fruitful field of the Jacobi structures by studying new Jacobi-like structures and some of their dual pairs. The aim is achieved in three steps. First, motivated by the active role played by linear algebra in differential geometry, both at the level of conceptualization (e.g. the construction of vector bundles equipped with various structures) and organization, we analyze some linear structures involved in modern geometric constructions, like Poisson, symplectic, contact, and Jacobi structures. Second, we introduce and investigate a new kind of Jacobi pair, namely a Jacobi pair with background, which, in particular, encompasses the twisted Jacobi notion. Third, we analyze from the global perspective (line bundle formulation) the previous Jacobi-like structure.

The thesis is structured on four chapters and a paragraph, as follows.

The paragraph contains the main conventions and notations adopted throughout the content.

Chapter 2 is dedicated to linear (algebraic) Jacobi structures and to linear contact dual pairs. The motivation of this chapter arises from the main goal of the present work as linear Jacobi structures offer valuable information on their corresponding smooth versions. Initially, we give a linear (algebraic) setting for Jacobi structures (Jacobi vector spaces) that comes from the pointwise perspective of a Jacobi pair. This is shown to be a specific instance of a more general one (L-Jacobi vector space) displayed by the same 'surgery' but in a Jacobi (line) bundle. In the remaining part of the chapter, we focus only on the Jacobi vector spaces and their relations with contact/locally conformal symplectic/Poisson structures. Then, we introduce and analyze the concept of linear contact pair. Concretely, starting with the orthogonality in contact vector spaces, we define the linear contact pair in a 'standard' manner via the contact orthogonality of the kernels corresponding to the 'legs'. We show that its symplectization is a linear (symplectic) dual pair. Also, it is shown that linear contact pairs enjoy of nice characteristic subspace correspondences and of Jacobi transversal correspondences. The original contribution contained in this chapter is based on [14, 15].

Chapter 3 is devoted to a new concept–*Jacobi structure with background*. Due to the special framework–trivial line bundle, such structures consist of pairs of geometric objects (one 2-vector field and one vector field) and are addressed as *Jacobi pairs with background*. The twisted Jacobi pairs and the Poisson structures with background are special cases of this new construct. Also, here, the completion of the category whose objects are the Jacobi manifolds with background (manifolds endowed with Jacobi pairs with background) is done. It is adapted and investigated the notions of Jacobi map and conformal Jacobi morphism [62]. Further, we prove that there is a one-to-one correspondence between Jacobi manifolds with background and homogeneous Poisson manifolds with background. Concerning the characteristic distribution associated with a Jacobi structure with background, we prove that it is completely integrable with its characteristic leaves consisting in either locally conformal symplectic manifolds with background or twisted contact manifolds. The chapter is concluded with twisted dual pairs in the symplectic and contact setting. We emphasize two results, one concerning the characteristic leaf correspondence and the other about "Poissonization" of a twisted contact dual pair. The original contribution to this chapter is contained in [16, 17, 18].

Chapter 4 is dedicated to the global (non-trivial) line bundle version of the Jacobi structure with background analyzed in the previous chapter. The strategy consists of four steps as follows. First, we make a trip in the realm of Lie and Jacobi algebroids, where we collect their standard characterizations [31, 32]. Unavoidable, this includes the Atiyah algebroid of the derivations of a line bundle [43]. For connecting the line-bundle formulations of the analyzed Jacobi bundles with the 'pairs', the previously mentioned results are done also in the trivial line bundle context [73]. Second, for self-consistency reasons, we shortly address the Jacobi bundles [56] and their characteristic distributions integrability [23, 73]. Third, we approach twisted Jacobi bundles. In the literature, this has been previously done only in the context of the trivial line bundle, i.e., in our language, twisted Jacobi pairs. Here, we collect the main results concerning transitive twisted Jacobi bundles and the integrability of twisted Jacobi bundles. Fourth, we introduce and analyze Jacobi bundles with background. It is shown that the trivial line bundle version of a Jacobi bundle with background is nothing but a Jacobi pair with background. Then, the analysis of transitive Jacobi bundles with background allows us to conclude that they are equivalent to either a locally conformal symplectic structure with background or a twisted conformal structure. Finally, by using the fact that locally, any Jacobi bundle with background is equivalent to a Jacobi pair with background, we sketch the proof of the integrability of Jacobi bundles with background. The original results contained in the present chapter are based on [16, 19, 20].

Conventions and notations

MAN. All manifolds in this work are assumed to be smooth and finite-dimensional. These are denoted by the underlying set, not specifying the smooth structure, i.e., the smooth manifold $(M, \mathcal{A}_M)$ is addressed as M. For a given manifold M, by $\mathcal{F}(M)$ we mean the $\mathbb{R}$-algebra of real smooth functions defined over M. Also, we denote by $\mathfrak{X}^\bullet(M)$ and $\Omega^\bullet(M)$ the graded commutative unital algebras of smooth multivector fields and smooth differential forms respectively,

$$\mathfrak{X}^\bullet(M) = \mathcal{F}(M) \oplus \mathfrak{X}^1(M) \oplus \mathfrak{X}^2(M) \oplus \cdots, \quad \Omega^\bullet(M) = \mathcal{F}(M) \oplus \Omega^1(M) \oplus \Omega^2(M) \oplus \cdots.$$

The algebras are dual to each other i.e. there exists the non-degenerate linear map

$$\langle \bullet, \bullet \rangle : \Omega^\bullet(M) \times \mathfrak{X}^\bullet(M) \to \mathcal{F}(M), \quad \langle \omega_1 \wedge \cdots \wedge \omega_p, X_1 \wedge \cdots \wedge X_p \rangle := \det\left(\langle \omega_i, X_j \rangle\right).$$

We denote by $[\bullet, \bullet]$ the Schouten-Nijenhuis bracket among multivector fields, structure that organizes the graded, graded commutative, associative and unital algebra $(\mathfrak{X}^\bullet(M), \wedge)$ as Gerstenhaber algebra. Let M_1 and M_2 be two smooth manifolds and $F \in \mathcal{C}^\infty(M_1, M_2)$ be a smooth map. We denote by F_* and F^* the tangent and the pull-back maps respectively

$$(F_* X) f := X(f \circ F), \quad \langle F^* \omega, X \rangle := \langle \omega, F_* X \rangle, \quad X \in \mathfrak{X}^1(M_1), \omega \in \Omega^1(M_2), f \in \mathcal{F}(M_2).$$

VB. Throughout this work only finite-dimensional vector bundles appear. A vector bundle is addressed as $E \to M$. Meanwhile, its $\mathcal{F}(M)$-module of sections is denoted by $\Gamma(E)$. The vector bundle morphisms between vector bundles with different base manifolds are expressed as pairs of maps $(F, \underline{F})$, where the first entry relate the total spaces while the second maps the base manifolds. When the vector bundles have the same base manifold then the vector bundle morphisms (F, id) are addressed only by F. For a given vector bundle morphism F between the vector bundles $E_1 \to M$ and $E_2 \to M$ the same symbol is used for the associated morphism of graded commutative unital algebras $F : \Gamma(\wedge^\bullet E_1) \to \Gamma(\wedge^\bullet E_2)$. In the algebras $\Gamma(\wedge^\bullet E)$ and $\Gamma(\wedge^\bullet E^*)$ that are dual to each other

$$\langle \bullet, \bullet \rangle : \Gamma(\wedge^\bullet E^*) \times \Gamma(\wedge^\bullet E) \to \mathcal{F}(M), \quad \langle \omega_1 \wedge \cdots \wedge \omega_p, \alpha_1 \wedge \cdots \wedge \alpha_p \rangle := \det\left(\langle \omega_i, \alpha_j \rangle\right),$$

we denote by i_P the right inner products in $\Gamma(\wedge^\bullet E^*)$ by homogeneous elements $P \in \Gamma(\wedge^\bullet E)$,

$$\langle i_P \omega, Q \rangle := \langle \omega, P \wedge Q \rangle$$

and by j_ω the left inner products in $\Gamma(\wedge^\bullet E)$ by homogeneous elements $\omega \in \Gamma(\wedge^\bullet E^*)$,

$$\langle \theta, j_\omega P \rangle := \langle \theta \wedge \omega, P \rangle.$$

Within the specified context, we display the isomorphisms

$$\Gamma(E) = (\Gamma(E^*))^*, \quad \wedge^p\Gamma(E) = \Gamma(\wedge^p E) = (\Gamma(\wedge^p E^*))^* = (\wedge^p\Gamma(E^*))^* = \wedge^p(\Gamma(E^*))^*,$$

in terms of which, sections in the vector bundle $\wedge^p E$, $P \in \Gamma(\wedge^p E)$, are also understood as multi-linear and skew-symmetric maps

$$P : \Gamma(E^*) \times \cdots \times \Gamma(E^*) \to \mathcal{F}(M), \quad P(\theta_1, \cdots, \theta_p) := \langle \theta_1 \wedge \cdots \wedge \theta_p, P\rangle.$$

The same interpretation we understand for sections in dual vector bundle $\wedge^p E^*$. From this perspective, we adopt the conventions from [55] regarding the 'musical' maps

$$\begin{aligned} {}^\sharp &: \Gamma(\wedge^2 E) \to \Gamma(E^* \otimes E), \quad P \mapsto P^\sharp, \quad P^\sharp\theta := -j_\theta P, \quad \theta \in \Gamma(E^*), \\ {}^\flat &: \Gamma(\wedge^2 E^*) \to \Gamma(E \otimes E^*), \quad \omega \mapsto \omega^\flat, \quad \omega^\flat X := -i_X\omega, \quad X \in \Gamma(E). \end{aligned}$$

Chapter 2

Linear (algebraic) Jacobi structures and dual pairs

Motivated by the active role played by linear algebra in differential geometry, both at the level of conceptualization (e.g. the construction of vector bundles equipped with various structures) and organization, in this chapter, we are going to discuss about some linear structures involved in modern geometric constructions, like Poisson, symplectic, contact, and Jacobi structures. Concretely, in the present chapter, we study those properties of linear algebraic nature which sit at the foundation of Poisson/symplectic/contact/Jacobi geometries. Of course, the linear aspects do not 'see' the smooth part, losing important features (e.g. brackets), but display many interesting features (e.g. dual pairs and transversals). The present chapter represents linear 'prolegomena' for the next part of the thesis, and, at the same time, it gives a consistent development of a linear version of the contact dual pair introduced in [73].

We introduce the broadest definition of contact and Jacobi structures[1], but we scrutinize the linear versions of contact/Jacobi structures with trivialized line bundles (i.e. cooriented). The linear Poisson structures [51] on a vector space V are bi-vectors $\pi \in \Lambda^2 V$, while the symplectic structures on V (non-degenerate 2-forms in $\Lambda^2 V^*$) are their non-degenerate versions. The linear versions of Jacobi structures with trivialized line bundles are simply pairs (π, E), with bi-vector $\pi \in \Lambda^2 V$ and vector $E \in V$, while the non-degenerate Jacobi structures are the contact structures. The broadest definition of Jacobi structures, which are addressed as L(line)-Jacobi structures [15], makes use of an L-vector space that consists of a line, L, and a short exact sequence of vector spaces,

$$\mathbb{D} : 0 \longrightarrow \mathbb{R} \longrightarrow D \longrightarrow V \longrightarrow 0.$$

Within this context, a L-Jacobi structure is done by an L-valued 2-form. When $L = \mathbb{R}$ and $\mathbb{D}$ admits a split, the L-Jacobi structure reduces to a Jacobi one.

The linear version of a contact structure has been introduced in [52]: a (contact) hyperplane H endowed with a V/H-valued non-degenerate skew-symmetric (curvature) 2-form, ω_H. This can be captured in the L-Jacobi framework whenever the L-valued 2-form is non-degenerate. This precisely means that the L-contact structure is a broader concept that

[1]We are indebted to L. Vitagliano and A. Tortorella for helping us with pertinent remarks concerning the most general context for Jacobi structures.

encompasses [15] Loose's contact one. The cooriented counterpart involves the choice of a (Reeb) vector E outside H, so that the previously mentioned non-degenerate skew-symmetric (curvature) 2-form becomes $\omega_H \in \Lambda^2 H^*$. An equivalent definition of linear contact structures, which reminds of the concept of contact differential form, involves a pair (θ, ω) with 1-form θ and 2-form ω which plays the role of the differential of the contact form θ, with the property that the top degree form $\theta \wedge \omega^m$ is non-zero.

In the linear Poisson setting the Hamiltonian vectors are associated to covectors, namely $X_\alpha = \pi^\sharp \alpha$, but for linear Jacobi structures they are of the form $X_A = aE + \pi^\sharp \alpha$ for $A = (\alpha, a) \in V^* \times \mathbb{R}$. The characteristic subspace of the Jacobi vector space V, generated by the Hamiltonian vectors, inherits in a natural way either a linear contact structure (if odd dimensional), or a linear version of the locally conformally symplectic (lcs) structure (if even dimensional). Thus, totally analogous to the differential geometric setting in [24], the transitive linear Jacobi structures are either contact or locally conformally symplectic. The pairs $A = (\alpha, a)$, with $a \in \mathbb{R}^\times$ also serve as conformal factors, both for conformal equivalence of linear Jacobi structures and for conformal Jacobi maps.

In this chapter we touch also several other aspects of linear Jacobi structures, like Poissonization, coisotropy, product structures, and especially transversals. In the literature the Poisson/Jacobi transversals can be also found under the name of cosymplectic submanifolds [83] or Poisson/Jacobi submanifolds of the second kind [82, 24, 54]. We define the linear Jacobi transversals as subspaces that admit a direct sum decomposition $V = X \oplus \pi^\sharp X^\circ$. They inherit in a natural way Jacobi structures. The Jacobi transversals of contact/lcs vector spaces are their contact/lcs subspaces.

Our main focus is on the linear version of the contact dual pairs in [73]. The symplectic dual pair [82] has a linear counterpart that involves a symplectic vector space and a pair of Poisson vector spaces [11], namely it is a pair of linear Poisson maps

$$\begin{array}{ccccc} & & (W, \omega) & & \\ & \swarrow^{\psi_1} & & \searrow^{\psi_2} & \\ (W_1, \pi_1) & & & & (W_2, \pi_2) \end{array} \tag{2.1}$$

that satisfy the orthogonality condition $(\operatorname{Ker} \psi_1)^\omega = \operatorname{Ker} \psi_2$. Let V_1, V_2 be a pair of Jacobi vector spaces equipped with a pair of conformal Jacobi maps defined on the same contact vector space V,

$$\begin{array}{ccccc} & & (V, \omega_H, E) & & \\ & \swarrow^{\varphi_1} & & \searrow^{\varphi_2} & \\ (W_1, \pi_1) & & & & (W_2, \pi_2) \end{array} \tag{2.2}$$

with conformal factors $A_i = (\alpha_i, a_i) \in V^* \times \mathbb{R}^\times$. They form a linear contact dual pair if the following conditions are satisfied:

1. $X_{A_1} \in \operatorname{Ker} \varphi_2$, $X_{A_2} \in \operatorname{Ker} \varphi_1$,

2. $(H \cap \operatorname{Ker} \varphi_1)^{\omega_H} = H \cap \operatorname{Ker} \varphi_2$ (by 1., the subspaces H and $\operatorname{Ker} \varphi_i$ intersect transversally),

3. $\pi(\alpha_1, \alpha_2) = \alpha_2(X_{A_1}) - \alpha_1(X_{A_2})$,

with (π, E) the induced Jacobi structure on the contact vector space V.

The symplectization of a linear contact dual pair (2.2) is a linear symplectic dual pair (3.97). The contact dual pair is called full if the maps φ_1, φ_2 are surjective, in which case there exists a natural correspondence between the contact/lcs structures on the characteristic subspaces of the Jacobi vector spaces V_1 and V_2, as shown in Theorem 2.2.14. With two linear Jacobi transversals $X_1 \subseteq V_1$ and $X_2 \subseteq V_2$ in a given linear contact dual pair (2.2) on V, we build in Theorem 2.3.17 a new contact dual pair on $\varphi_1^{-1}(X_1) \cap \varphi_2^{-1}(X_2)$. A similar result holds also for two linear Poisson transversals in a linear symplectic dual pair (3.97).

The differential geometric concept of Poisson transversal was linked to symplectic dual pairs in [27]. The counterpart for Jacobi transversals and contact dual pairs in a differential geometric setting will be the subject of a future work.

The present chapter is organized as follows. Initially, we give a linear (algebraic) setting for Jacobi structures (Jacobi vector spaces) that comes from the pointwise perspective of a Jacobi pair. This is shown to be a specific instance of a more general one (L-Jacobi vector space) displayed by the same 'surgery' but in a Jacobi (line) bundle. In the remaining part of the chapter, we focus only on the Jacobi vector spaces and their relations with contact/locally conformal symplectic/Poisson structures. Then, we introduce and analyze the concept of linear contact pair. Concretely, starting with the orthogonality in contact vector spaces, we define the linear contact pair in a 'standard' manner via the contact orthogonality of the kernels corresponding to the 'legs'. We show that its symplectization is a linear (symplectic) dual pair. Also, it is shown that linear contact pairs enjoy nice characteristic subspace correspondences and Jacobi transversal correspondences.

The original contribution contained in this chapter is based on [14, 15].

2.1 A linear algebraic setting for Jacobi structures

In this section we put contact and Jacobi structures in a linear algebraic setting. In view of this, let's remember what a smooth Jacobi structure means (for more details see Chapter 4, Section 4.2). Initially defined by means of Jacobi pairs [33], Jacobi manifolds can also be defined on non-trivial line bundles [32, 56]. Let M be a smooth manifold and $L \to M$ be a line bundle. A Jacobi structure on the given line bundle is an $\mathbb{R}$-Lie algebra structure on the module of smooth sections $\Gamma(L)$,

$$\{\cdot, \cdot\} : \Gamma(L) \times \Gamma(L) \to \Gamma(L),$$

which, in addition, is a first-order differential operator in each entry [56], i.e.

$$\{\lambda, f\mu\} = X_\lambda(f)\mu + f\{\lambda, \mu\}, \quad \lambda, \mu \in \Gamma(L), \quad f \in \mathcal{F}(M).$$

Previously, X_λ is the Hamiltonian vector field on M, which is completely determined by $\{\cdot, \cdot\}$ and λ. The bracket $\{\cdot, \cdot\}$, called the *Jacobi bracket*, can be equivalently encoded in a vector bundle map

$$\hat{J} : \wedge^2 J^1 L \to L$$

satisfying an appropriate integrability condition, where J^1L denotes the first jet bundle of L. The manifold M equipped with a Jacobi structure $(L \to M, \{\cdot,\cdot\})$ is called a Jacobi manifold. An 'exact surgery' in a Jacobi manifold at one single point $x \in M$ reveals:

C1) a 1-dimensional vector space L_x;

C2) a short exact sequence of vector spaces $0 \to \mathbb{R} \to D_xL \to T_xM \to 0$;

C3) an L_x-valued 2-form $\widehat{\omega}_x : \Lambda^2 D_xL \to L_x$.

When $L = \mathbb{R}_M$ is the trivial line bundle over M, the definition of a Jacobi bracket boils down to a pair (Π, E) (in this trivial line bundle context $\hat{J}$ reduces to $\hat{J} = \Pi - E\wedge 1$) consisting of a bivector field Π and a vector field E satisfying appropriate integrability conditions. In this trivial line bundle context, an 'exact surgery' in a Jacobi manifold displays the bivector $\Pi_x \in \wedge^2 T_xM$ and the vector $E_x \in T_xM$.

A definition that captures the linear version of non-cooriented contact structures has been given by Loose:

Definition 2.1.1. [52] A *contact structure* on a vector space V is given by a hyperplane $H \subset V$ endowed with a skew-symmetric non-degenerate bilinear form $\omega_H : H \times H \to V/H$.

In the light of the previous pointwise 'surgery' in a Jacobi *line* bundle, we can exhibit [15] the notion of L-Jacobi structure as follows. Let L be a line (i.e. a 1-dimensional real vector space) and $\mathbb{V}$ be a short exact sequence of real vector spaces

$$0 \longrightarrow \mathbb{R} \xrightarrow{\ i\ } \hat{V} \xrightarrow{\ \sigma\ } V \longrightarrow 0. \tag{2.3}$$

The pair $(L, \mathbb{V})$ will be called an *L-vector space*. By considering the dual of (2.3)

$$0 \longrightarrow V^* \xrightarrow{\ \sigma^*\ } \hat{V}^* \xrightarrow{\ i^*\ } \mathbb{R} \longrightarrow 0, \tag{2.4}$$

followed the tensor product with the line L, it results the short exact sequence

$$0 \longrightarrow V^* \otimes L \longrightarrow \hat{V}^* \otimes L \longrightarrow L \longrightarrow 0, \tag{2.5}$$

which allows the introduction of the L-Jacobi vector space concept.

Definition 2.1.2. [15] An *L-Jacobi structure* on an L-vector space $(L, \mathbb{V})$ is an L-valued 2-form

$$\tilde{\pi} : \wedge^2 \tilde{V} \to L, \tag{2.6}$$

where we used the notation

$$\tilde{V} := \hat{V}^* \otimes L. \tag{2.7}$$

If in addition the 2-form (2.6) is non-degenerate, we talk about an *L-contact structure.*

We show that the L-Jacobi structure encompasses the contact one because L-contact structures (i.e., non-degenerate L-Jacobi structures) are nothing but the contact ones. Indeed, if $\tilde{\pi}$ is a non-degenerate L-Jacobi structure on an L-vector space $(L, \mathbb{V})$, i.e. the linear map

$$\tilde{\pi}^\sharp : \tilde{V} \to \tilde{V}^* \otimes l, \quad \tilde{\pi}^\sharp(\alpha)(\beta) := \tilde{\pi}(\alpha \wedge \beta) \tag{2.8}$$

is non-degenerate, we can define its inverse, $\tilde{\pi}^\flat$. Further, the isomorphism $\tilde{V}^* \otimes L = \hat{V}$ exhibits the L-valued 2-form

$$\tilde{\omega} : \wedge^2 \hat{V} \to L, \quad \tilde{\omega}(x \wedge y) := \tilde{\pi}(\tilde{\pi}^\flat(x) \wedge \tilde{\pi}^\flat(y)), \tag{2.9}$$

which is non-degenerate as $\tilde{\omega}^\flat = \tilde{\pi}^\flat$. If we denote by $\mathbb{I} := i(1)$, then, via the exact sequence (2.5) it results that $\tilde{\omega}$ determines both to a nontrivial L-valued 1-form on V, $\theta \in V^* \otimes L$

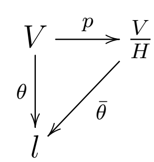
$$\theta \circ \sigma = \tilde{\omega}^\flat(\mathbb{I}) \tag{2.10}$$

and to an L-valued 2-form on $H := \operatorname{Ker}\theta$

$$\bar{\omega}_H : \wedge^2 H \to L, \quad \bar{\omega}_H(\sigma(X) \wedge \sigma(Y)) := \tilde{\omega}(X \wedge Y), \tag{2.11}$$

which is non-degenerate.

With these specifications at hand, by means of the universality of quotient space V/H, the original line L is isomorphic to the quotient space, i.e. there exists the unique linear map $\bar{\theta}$ that makes commutative the diagram



This isomorphism allows the identification of contact structure

$$\omega_H := \bar{\theta}^{-1} \circ \bar{\omega}_H. \tag{2.12}$$

To conclude with, we have shown that contact structures are encompassed by L-Jacobi structures. The previous construction stresses that the correspondence between L-contact and contact *is onto, but not one-to-one.*

2.1.1 Linear contact structures

A cooriented contact structure on a $(2m+1)$-dimensional differentiable manifold can be defined as a hyperplane field given by the kernel of a differential 1-form θ that satisfies one of the following two equivalent conditions:

1. the top degree form $\theta \wedge (d\theta)^m \neq 0$,

2. $d\theta$ is non-degenerate on $\operatorname{Ker}\theta$.

The first condition can be transferred to the linear setting as follows:

Definition 2.1.3. A *contact structure* on a vector space V with $\dim V = 2m+1$ is a pair (θ, ω) with $\theta \in V^*$ and $\omega \in \Lambda^2 V^*$, such that $\theta \wedge \omega^m \neq 0$.

The *contact hyperplane* H and *cocontact line* L, given by

$$H := \operatorname{Ker}\theta, \quad L := \operatorname{Ker}\omega^\flat, \tag{2.13}$$

satisfy $V = H \oplus L$. There exists a unique *Reeb vector* $E \in V$ that satisfies

$$i_E\theta = 1, \quad i_E\omega = 0, \tag{2.14}$$

thus $L = \langle E \rangle$. The restriction of ω to the hyperplane H is a non-degenerate 2-form $\omega_H \in \Lambda^2 H^*$, called the *curvature form* (it plays the role of $\mathrm{d}\theta|_{\operatorname{Ker}\theta}$ from the coorientable smooth situation).

The second condition yields an equivalent definition of a contact structure in the linear setting:

Definition 2.1.4. A *contact structure* on a vector space V is a pair (ω_H, E) consisting of a non-degenerate form $\omega_H \in \Lambda^2 H^*$ on a hyperplane $H \subset V$ and a vector $E \in V \setminus H$.

This definition is the cooriented version of the more general Definition 2.1.1 due to Loose. The choice of a nonzero vector $E \in V \setminus H$ permits the identification of the cocontact line $L = V/H$ with the line generated by E, and the curvature form becomes a non-degenerate skew-symmetric bilinear form $\omega_H \in \Lambda^2 H^*$, as in Definition 2.1.4.

Proposition 2.1.5. *The two definitions 2.1.3 and 2.1.4 are equivalent.*

Proof. We have seen how to pass from Definition 2.1.3 to Definition 2.1.4. For the way back, we assign to every (ω_H, E) as in Definition 2.1.4 a pair (θ, ω) as in Definition 2.1.3. These are the unique $\theta \in V^*$ subject to the conditions

$$H = \operatorname{Ker}\theta, \quad \theta(E) = 1, \tag{2.15}$$

and the 2-form $\omega \in \Lambda^2 V^*$ defined by

$$\omega := p_H^* \omega_H \in \Lambda^2 V^*, \tag{2.16}$$

where p_H denotes the projection on H parallel to the Reeb vector E:

$$p_H : V \to H, \quad p_H(x) := x - \theta(x)E. \tag{2.17}$$

Noticing that $\operatorname{rank}\omega = 2m$, we deduce that $\theta \wedge \omega^m \neq 0$, by using Lepage's Theorem [51, Proposition 2.5], which is the condition in Definition 2.1.3. Moreover, $\omega|_H = \omega_H$ and $i_E\omega = 0$ ensures that from (θ, ω) we regain the structure (ω_H, E) we started with. □

Remark 2.1.6 (Symplectization of contact structures)**.** To any contact structure (θ, ω) on the vector space V we associate the non-degenerate 2-form[2]

$$\hat{\omega} \in \Lambda^2 (V \oplus \mathbb{R})^*, \quad \hat{\omega} := \omega + \theta \wedge 1, \tag{2.18}$$

[2] In reference [15], we considered an additional factor in definition (2.18), i.e. $\hat{\omega} := \omega - \theta \wedge 1$.

called the *symplectization of the linear contact structure*. Previous definition is consistent in the light of isomorphism

$$\wedge^p(V \oplus \mathbb{R})^* = \wedge^p V^* \oplus \wedge^{p-1} V^*, \quad 1 \leq p \leq \dim V.$$

exhibited by a natural pairing (see below (2.20)).

For any $x \oplus t \in V \oplus \mathbb{R}$ we have that

$$i_{x \oplus t}\hat{\omega} = (i_x\omega - t\theta) \oplus \theta(x) \in V^* \oplus \mathbb{R}, \tag{2.19}$$

so $x \oplus t$ lies in $\operatorname{Ker}\hat{\omega}$ if and only if $i_x\omega = t\theta$ and $\theta(x) = 0$. The first identity contracted with E implies that $t = 0$. Thus $i_x\omega = 0$, and together with $i_x\theta = 0$ we get that $i_x(\theta \wedge \omega^m) = 0$, hence $x = 0$ too. We conclude that the form $\hat{\omega}$ is non-degenerate, hence symplectic.

Alternatively, we see that $\hat{\omega}^{m+1} = \theta \wedge \omega^m \wedge 1 \neq 0$ on the $2m+2$ dimensional space $V \oplus \mathbb{R}$.

2.1.2 Jacobi vector spaces and their Poissonizations

A linear Poisson structure on a vector space V is a bi-vector $\pi \in \wedge^2 V$ [51]. Linear symplectic structures $\omega \in \Lambda^2 V^*$ admit canonical linear Poisson structures, given by $\pi^\sharp = (\omega^\flat)^{-1} : V^* \to V$, where $\pi^\sharp(\eta) = i_\eta\pi$ for all $\eta \in V^*$ and $\omega^\flat(x) := -i_x\omega$ for all $x \in V$ [55].

Definition 2.1.7. A *linear Jacobi structure* on a vector space V is a pair (π, E) with

$$\pi \in \Lambda^2 V, \quad E \in V.$$

At this point, it is natural to ask about the relation between the L-Jacobi and Jacobi concepts. If in Definition 2.1.2 we consider the L-vector space $(\mathbb{R}, \mathbb{V})$, with $\mathbb{V}$ admitting a split (say $j : V \to \hat{V}$), then there exists the isomorphism $\hat{V} = V \oplus \mathbb{R}$ and the L-Jacobi structure becomes $\tilde{\pi} \in \wedge^2(V \oplus \mathbb{R})$. On the other hand, by considering the pairing

$$\langle \bullet, \bullet \rangle : (V \oplus \mathbb{R}) \times (V^* \oplus \mathbb{R}) \to \mathbb{R}, \quad \langle x \oplus t, \alpha \oplus s \rangle := \alpha(x) + st, \tag{2.20}$$

we can identify $(V \oplus \mathbb{R})^* = V^* \oplus \mathbb{R}$ which results in the isomorphisms

$$\wedge^p(V \oplus \mathbb{R}) = \wedge^p V \oplus \wedge^{p-1} V, \quad 1 \leq p \leq \dim V.$$

The last result allows the identification $\tilde{\pi} = \pi \oplus E$ which makes transparent the linear Jacobi structure (π, E).

In the next chapters we shall sketch a similar connection between Jacobi(-like) line bundles and Jacobi(-like) pairs.

Example 2.1.8. By its very definition, it is clear that a Poisson vector space is a Jacobi one exhibiting a trivial Reeb vector E.

Example 2.1.9. A linear algebraic version of a locally conformally symplectic structure, called a *linear lcs structure* on a vector space V is a pair (ω, λ) consisting of a non-degenerate skew-symmetric 2-form $\omega \in \Lambda^2 V^*$ and a 1-form $\lambda \in V^*$, called the *Lee form*. It has a natural underlying Jacobi structure given by $\pi^\sharp = (\omega^\flat)^{-1}$ and $E = \pi^\sharp\lambda$. In particular $V = \operatorname{Im}\pi^\sharp$ is even dimensional.

Example 2.1.10. Every contact structure (ω_H, E) on a given vector space V generates a Jacobi structure (π, E) on V. The bi-vector is defined by

$$\pi^\sharp : V^* \to V, \quad \pi^\sharp := \iota_H(\omega_H^\flat)^{-1}\iota_H^*, \tag{2.21}$$

where $\iota_H : H \to V$ denotes the inclusion. The relation (2.21) can be rewritten as

$$\pi(\beta, \gamma) = \omega_H(\pi^\sharp\beta, \pi^\sharp\gamma), \quad \forall \beta, \gamma \in V^*. \tag{2.22}$$

Because $\operatorname{Im} \iota_H^* = H^*$ and $\operatorname{Im} \pi^\sharp = H$, we have that $V = H \oplus \langle E \rangle = \operatorname{Im} \pi^\sharp \oplus \langle E \rangle$.

Definition 2.1.11. A subspace W of a contact vector space (V, ω_H, E) is called a *contact subspace* if W is transverse to H and $\omega_H|_{H\cap W}$ is non-degenerate. A subspace W of an lcs vector space (V, ω, λ) is called a *lcs subspace* if $\omega|_W$ is non-degenerate.

The subspace W inherits in a natural way a contact/lcs structure with induced Jacobi structure (π_W, E_W). Indeed, denoting by

$$W^\circ := \{\alpha \in V^* : W \subseteq \operatorname{Ker} \alpha\},$$

the annihilator associated with the subspace $W \subseteq V$, the invoked structure is constructed with the help of the projection p_W on the first factor in the direct sum decomposition

$$V = W \oplus \pi^\sharp W^\circ, \tag{2.23}$$

namely $E_W = p_W(E)$ and $\pi_W = (\Lambda^2 p_W)(\pi)$ (see also Corollary 2.3.12).

The decomposition (2.23) is clear in the lcs case, since $W^\omega = \pi^\sharp W^\circ$, the orthogonal of W with respect to the non-degenerate 2-form ω. For the contact case it follows from the following:

Lemma 2.1.12. *In a contact vector space (V, ω_H, E), any subspace U transverse to H satisfies*

$$(H \cap W)^{\omega_H} = \pi^\sharp(W^\circ). \tag{2.24}$$

Proof. By means of the equality $(H \cap W)^{\omega_H} = \omega_H^\sharp((H \cap W)^\circ)$, combined with the definition $\pi^\sharp = \omega_H^\sharp \iota_H^*$ of the Jacobi bi-vector associated with a given contact structure, the identity (2.24) is equivalent to

$$(H \cap W)^\circ = \iota_H^* (W^\circ),$$

where the annihilator lies inside H^*. The inclusion $\iota_H^*(W^\circ) \subseteq (W \cap H)^\circ$ is immediate. It is an equality by a dimension count. From the transversality $H + W = V$ follows that $\theta \notin W^\circ$. Since $\operatorname{Ker} i_H^* = \langle \theta \rangle$, we get $\dim i_H^*(W^\circ) = \dim W^\circ = \dim V - \dim W = \dim H - \dim(H \cap W) = \dim(H \cap W)^\circ$. □

In the contact case, the curvature form is the restriction of ω_H to the contact hyperplane $H_W = H \cap W$. In the lcs case, both the non-degenerate 2-form and the Lee form are obtained by restriction: $\omega_W = \omega|_W$ and $\lambda_W = \lambda|_W$.

Next we present the linear version of the one-to-one correspondence between Jacobi structures and homogeneous Poisson structures over smooth manifolds [56, 76].

Definition 2.1.13. Let (V, π, E) be a Jacobi vector space. Then the vector space $V \oplus \mathbb{R}$ can be canonically endowed with a Poisson structure[3]

$$\hat{\pi} := \pi + E \wedge 1, \tag{2.25}$$

called the *Poissonization* of the Jacobi structure.

Remark 2.1.14. *At this point, it is noteworthy that the Poisson structure on the Poisson vector space* $(V \oplus \mathbb{R}, \hat{\pi})$*,* $\hat{\pi}$*, is just an L-Jacobi structure on the L-vector space* $(\mathbb{R}, \mathbb{V})$

$$0 \longrightarrow \mathbb{R} \xrightarrow{i_2} V \oplus \mathbb{R} \xrightarrow{p_1} V \longrightarrow 0.$$

Proposition 2.1.15. *The symplectization* (2.18) *of a contact structure leads to the same result as the Poissonization* (2.25) *of its underlying Jacobi structure offered by Example 2.1.10.*

Proof. We have to show that the bi-vector $\hat{\pi} := \pi + E \wedge 1$ is the inverse of the 2-form $\hat{\omega} = \omega + \theta \wedge 1$. Similarly to (2.19) we get that

$$\hat{\pi}^\sharp(\eta \oplus s) = (\pi^\sharp(\eta) - sE) \oplus \eta(E), \quad \eta \in V^*, s \in \mathbb{R}, \tag{2.26}$$

and a direct computation yields for all $x \in V$ and $t \in \mathbb{R}$:

$$\begin{aligned} \langle (\eta \oplus s), \hat{\pi}^\sharp \hat{\omega}^\flat (x \oplus t) \rangle &= \langle (i_x\omega - t\theta) \oplus \theta(x), (\pi^\sharp(\eta) - sE) \oplus \eta(E) \rangle \\ &= -s\omega(x, E) - t\theta(\pi^\sharp\eta) + \omega(x, \pi^\sharp\eta) + st + \eta(E)\theta(x) \\ &= \eta(x) + st = \langle \eta \oplus s, x \oplus t \rangle. \end{aligned}$$

At step three we used $i_E\omega = 0$ and $\operatorname{Im}\pi^\sharp = H = \operatorname{Ker}\theta$, together with the identity

$$\omega(x, \pi^\sharp\eta) = \omega_H(p_H(x), \pi^\sharp\eta) = \eta(p_H(x)) = \eta(x) - \eta(E)\theta(x).$$

We conclude that $\hat{\pi}^\sharp\hat{\omega}^\flat = 1_{V \oplus \mathbb{R}}$, thus the bi-vector $\hat{\pi}$ is the inverse of the 2-form $\hat{\omega}$. □

2.1.3 Non-degenerate Jacobi structures

The non-degenerate Poisson structures are the symplectic ones. A natural way to introduce the non-degenerate Jacobi structures is via Poissonization: (π, E) is *non-degenerate Jacobi* if and only if $\hat{\pi} = \pi + E \wedge 1$ is non-degenerate Poisson.

Lemma 2.1.16. *The Poisson structure* $\hat{\pi}$ *on* $V \oplus \mathbb{R}$ *is non-degenerate if and only if*

$$V = \operatorname{Im}\pi^\sharp \oplus \langle E \rangle. \tag{2.27}$$

Proof. We analyze the injectivity of $\hat{\pi}^\sharp$, given in (2.26), and we get that the Poisson structure $\hat{\pi}$ is non-degenerate if and only if

$$E \notin \operatorname{Im}\pi^\sharp \quad \text{and} \quad E^\circ \cap \operatorname{Ker}\pi^\sharp = 0. \tag{2.28}$$

The annihilator of the second formula in (2.28) gives $\langle E \rangle + \operatorname{Im}\pi^\sharp = V$, hence the conclusion. □

[3] In [15] an extra-sign has been considered, i.e. $\hat{\pi} := \pi - E \wedge 1$

Definition 2.1.17. The Jacobi structure (π, E) on V is called *non-degenerate* if it satisfies the identity (2.27).

Due to the fact that the range of $\pi^\sharp$ is even-dimensional, it is clear from (2.27) that non-degenerate Jacobi vector spaces are odd-dimensional. The Example 2.1.10 ensures that the contact structures are non-degenerate Jacobi structures. The converse also holds.

Proposition 2.1.18. *If (π, E) is a non-degenerate Jacobi structure on V, then there exists a unique contact structure on V with induced Jacobi structure (π, E).*

Proof. Starting with a Jacobi structure (π, E) that enjoys (2.27), we define a contact structure (ω_H, E) with hyperplane $H := \operatorname{Im}\pi^\sharp$ and 2-form

$$\omega_H(\pi^\sharp\beta, \pi^\sharp\gamma) = \pi(\beta, \gamma), \quad \forall \beta, \gamma \in V^*. \tag{2.29}$$

This means that $\omega_H(x, \pi^\sharp\gamma) = \gamma(x)$ for all $x \in H$, so ω_H is well-defined by $\operatorname{Ker}\pi^\sharp = H^\circ$. It is non-degenerate since $\operatorname{Ker}\omega_H = \pi^\sharp(H^\circ) = 0$. We conclude that (ω_H, E) is indeed a contact structure on V. The identity (2.22) ensures that the underlying Jacobi structure is the given Jacobi structure (π, E). □

2.1.4 Characteristic subspaces and transitivity

For a Poisson vector space (V, π), the Hamiltonian vectors $X_\alpha = \pi^\sharp\alpha$ are associated to covectors $\alpha \in V^*$. They generate the even dimensional characteristic subspace $C = \operatorname{Im}\pi^\sharp$, which inherits a non-degenerate Poisson structure $\pi_C \in \Lambda^2 C$, hence a symplectic structure $\omega_C \in \Lambda^2 C^*$ [51, Proposition 4.6]:

$$\omega_C(\pi^\sharp\beta, \pi^\sharp\gamma) = \pi(\beta, \gamma), \quad \forall \beta, \gamma \in V^*. \tag{2.30}$$

Let (π, E) be a Jacobi structure on the vector space V. To each pair $A = (\alpha, a) \in V^* \times \mathbb{R}$ we assign a *Hamiltonian vector*

$$X_A := aE + \pi^\sharp\alpha. \tag{2.31}$$

To establish the analogy with Hamiltonian vector fields X_f on Jacobi manifolds, the number a stands for the value of the Hamiltonian function f, while the covector α stands for its differential df. Remember here that, according with the preamble of this section, a Jacobi structure on the trivial line bundle $\mathbb{R}_M$ stands in a bi-differential operator $\hat{J} = \Pi - E \wedge 1$, $\hat{J} : \wedge^2 J^1\mathbb{R}_M \to \mathbb{R}_M$. Due to the vector bundle isomorphism $J^1\mathbb{R}_M = T^*M \oplus \mathbb{R}_M$, the Hamiltonian vector field associated with a smooth function $f \in \mathcal{F}(M)$ (via its prolongation to the first-order jet bundle $j^1 f = \mathrm{d}f \oplus f$) is consistently defined by

$$(\mathrm{pr}_1 \circ \hat{J}^\sharp)(\mathrm{d}f \oplus f) = \Pi^\sharp \mathrm{d}f + fE.$$

Previously, we denoted by pr_1 the projection on the first factor $\mathrm{pr}_1 : TM \oplus \mathbb{R}_M \to TM$, which is a vector bundle map.

Remark 2.1.19. At this stage, it is useful to express the Hamiltonian vectors associated with a given contact structure (θ, ω) on V. Similarly to the differential geometric setting, for each pair $A = (\alpha, a) \in V^* \times \mathbb{R}$ we define the *contact Hamiltonian vector* $X_A \in V$ subject to the conditions

$$i_{X_A}\theta = a, \quad i_{X_A}\omega = -\alpha + \alpha(E)\theta. \tag{2.32}$$

Since $\theta \wedge \omega^m \neq 0$, the two conditions single out a unique vector.

It coincides with the Hamiltonian vector (2.31) for the associated linear Jacobi structure. The first condition in (2.32) is satisfied because $i_E\theta = 1$ and $\operatorname{Im}\pi^\sharp = \operatorname{Ker}\theta$. For the second condition we compute

$$-i_{X_A}\omega \overset{(2.14)}{=} \omega^\flat\pi^\sharp\alpha \overset{(2.21)}{=} \omega^\flat\iota_H(\omega_H^\flat)^{-1}\iota_H^*\alpha \overset{(2.16)}{=} p_H^*\omega_H^\flat p_H\iota_H(\omega_H^\flat)^{-1}\iota_H^*\alpha = p_H^*\iota_H^*\alpha \overset{(2.17)}{=} \alpha-\alpha(E)\theta.$$

Definition 2.1.20. The Hamiltonian vectors of a Jacobi vector space generate the subspace

$$C = \operatorname{Im}\pi^\sharp + \langle E\rangle, \tag{2.33}$$

called the *characteristic subspace.*

The characteristic subspace C inherits a Jacobi structure (π_C, E) in a canonical way. The bi-vector $\pi_C \in \Lambda^2 C$ is given by

$$\pi_C\left(\iota_C^*\beta, \iota_C^*\gamma\right) = \pi\left(\beta,\gamma\right), \quad \forall \beta,\gamma \in V^*, \tag{2.34}$$

where the surjective map $\iota_C^* : V^* \to C^*$ is associated with the inclusion $\iota_C : C \to V$. The inclusion $\operatorname{Ker}\iota_C^* = C^\circ \subseteq (\operatorname{Im}\pi^\sharp)^\circ$ guarantees that the bi-vector π_C is well-defined.

By the same formula (2.30) as in the Poisson case, we define a non-degenerate 2-form ω_C on $\operatorname{Im}\pi^\sharp$. Now we can give the linear analogue of the characteristic leaf theorem:

Proposition 2.1.21.

1. *If $E \in \operatorname{Im}\pi^\sharp$, then the characteristic subspace $C = \operatorname{Im}\pi^\sharp$ is even dimensional and the induced Jacobi structure (π_C, E) is an lcs structure (ω_C, λ_C), with Lee form $\lambda_C = -i_E\omega_C$.*
2. *If $E \notin \operatorname{Im}\pi^\sharp$, then the characteristic subspace $C = \operatorname{Im}\pi^\sharp \oplus \langle E\rangle$ is odd dimensional and the induced Jacobi structure (π_C, E) is a contact structure with contact hyperplane $H = \operatorname{Im}\pi^\sharp$ and curvature form $\omega_H = \omega_C$.*

Definition 2.1.22. The Jacobi structure (π, E) on V is said to be *transitive* if the characteristic subspace C is the whole V, i.e.

$$V = \operatorname{Im}\pi^\sharp + \langle E\rangle. \tag{2.35}$$

By Proposition 2.1.21, the transitive Jacobi structures fall into one of the following two categories.

1. If $E \notin \operatorname{Im}\pi^\sharp$, then $V = \operatorname{Im}\pi^\sharp \oplus \langle E\rangle$ is odd dimensional with non-degenerate Jacobi structure. Proposition 2.1.18 ensures that the Jacobi structure comes from a contact structure on V.
2. If $E \in \operatorname{Im}\pi^\sharp$, then $V = \operatorname{Im}\pi^\sharp$ is even dimensional and $\pi^\sharp : V^* \to V$ is an isomorphism. Thus the Jacobi structure comes from an lcs structure (ω, λ) on V, where the non-degenerate 2-form ω is the inverse of π and the Lee form is $\lambda = \omega^\flat E$ (see Example 4.2.3).

2.1.5 Conformal equivalence of Jacobi structures

The concept of conformally equivalent Jacobi structures on manifolds [56, 76] can also be adapted to our linear framework.

Definition 2.1.23. Let V be a vector space, (π, E) a Jacobi structure on it, and $A := (\alpha, a) \in V^* \times \mathbb{R}^\times$. The Jacobi structure given by

$$\pi^{(A)} := a\pi, \quad E^{(A)} := aE + \pi^\sharp\alpha \tag{2.36}$$

is said to be A-*conformally equivalent* to (π, E).

We notice that $E^{(A)} = X_A$ the Hamiltonian vector on the Jacobi vector space V. The conformal equivalence preserves the transitivity property, as well as the non-degeneracy property of linear Jacobi structures. It also maintains the dimension, hence the type of characteristic subspace.

Remark 2.1.24. Combining the above definition with (2.25) we conclude that, if the Jacobi structures (π, E) and $\left(\pi^{(A)}, E^{(A)}\right)$ are conformally-related, then their Poissonizations (on $V \oplus \mathbb{R}$) satisfy

$$\hat{\pi}^{(A)} = a\hat{\pi} + \pi^\sharp\alpha \wedge 1, \tag{2.37}$$

so the Poissonizations are conformally related if and only if $\alpha \in \operatorname{Ker} \pi^\sharp$.

Below we express the conformal equivalence for contact and lcs structures. First we examine two linear contact structures of two conformally equivalent transitive Jacobi structures (π, E) and $(\pi^{(A)}, E^{(A)})$ on an odd dimensional vector space V. They have the same contact hyperplane $H = \operatorname{Im} \pi^\sharp$, since $\pi^{(A)\sharp} = a\pi^\sharp$ with $a \neq 0$. It follows that the contact forms differ by a constant factor. We must have $\theta^{(A)} = a^{-1}\theta$ because $i_{E^{(A)}}\theta = ai_E\theta + i_{\pi^\sharp\alpha}\theta = a$. The curvature form is obtained with the formula (2.29):

$$a^2\omega_H^{(A)}(\pi^\sharp\beta, \pi^\sharp\gamma) = \omega_H^{(A)}((\pi^{(A)})^\sharp\beta, (\pi^{(A)})^\sharp\gamma) = \pi^{(A)}(\beta, \gamma) = a\pi(\beta, \gamma) = a\omega_H(\pi^\sharp\beta, \pi^\sharp\gamma)$$

for all $\beta, \gamma \in V^*$, hence $\omega_H^{(A)} = a^{-1}\omega_H$.

Definition 2.1.25. The contact structure $(\omega_H^{(A)}, E^{(A)})$ with

$$\omega_H^{(A)} := a^{-1}\omega_H, \quad E^{(A)} := aE + \omega_H^\sharp\iota_H^*\alpha \tag{2.38}$$

is called A-*conformally equivalent* to the contact structure (ω_H, E).

The description of conformal equivalence in terms of (θ, ω) is: $\theta^{(A)} = a^{-1}\theta$ and $\omega^{(A)} = a^{-1}\omega - a^{-2}\alpha \wedge \theta$. (It is the formula expected from formally differentiating $\theta^{(A)} = a^{-1}\theta$.)

Next we describe the relation between the linear lcs structures of two conformally equivalent transitive Jacobi structures (π, E) and $(\pi^{(A)}, E^{(A)})$ on an even dimensional vector space V. It is clear that the non-degenerate 2-forms ω and $\omega^{(A)}$, inverses of the isomorphisms $\pi^\sharp$ and $\left(\pi^{(A)}\right)^\sharp = a\pi^\sharp$, are related by $\omega^{(A)} = a^{-1}\omega$. Now we can see that the Lee 1-forms are related by

$$\lambda^{(A)} = \left(\omega^{(A)}\right)^\flat E^{(A)} = a^{-1}\omega^\flat(aE + \pi^\sharp\alpha) = \omega^\flat E + a^{-1}\alpha = \lambda + a^{-1}\alpha.$$

(Here $a^{-1}\alpha$ plays the role of the logarithmic derivative of a.)

Definition 2.1.26. The linear lcs structure $(\omega^{(A)}, \lambda^{(A)})$ with

$$\omega^{(A)} = a^{-1}\omega, \quad \lambda^{(A)} = \lambda + a^{-1}\alpha$$

is called *A-conformally equivalent* to the lcs structure (ω, λ).

2.1.6 Morphisms of Jacobi vector spaces

In this section we are going to complete the category of Jacobi vector spaces with morphisms of Jacobi vector spaces. We start from the concept of morphism of Poisson vector spaces [51]. Recall that a morphism between two Poisson vector spaces (V, π) and (V', π') is a linear map $\varphi : V \to V'$ such that $(\Lambda^2\varphi)\pi = \pi'$, i.e. $\pi'(\beta', \gamma') = \pi(\varphi^*\beta', \varphi^*\gamma')$, with $\varphi^* : V'^* \to V^*$ denoting the dual of φ. An equivalent way to express this is $(\pi')^\sharp = \varphi\pi^\sharp\varphi^*$.

Definition 2.1.27. Let (V, π, E) and (V', π', E') be two Jacobi vector spaces. The linear map $\varphi : V \to V'$ is said to be a *Jacobi map* if

$$(\Lambda^2\varphi)\pi = \pi', \quad \varphi E = E'. \tag{2.39}$$

The connection between Jacobi structures on V and Poisson structures on $V \oplus \mathbb{R}$ via Poissonization (2.25) can be used to characterize the Jacobi morphisms in terms of Poisson morphisms. A special case of Proposition 2.1.29 ensures that the linear map $\varphi : V \to V'$ is a Jacobi morphism if and only if the linear map

$$\hat{\varphi} : V \oplus \mathbb{R} \to V' \oplus \mathbb{R}, \quad \hat{\varphi}(x \oplus t) := \varphi(x) \oplus t$$

is a Poisson morphism between the Poissonizations $(V \oplus \mathbb{R}, \hat{\pi})$ and $(V' \oplus \mathbb{R}, \hat{\pi}')$.

Definition 2.1.28. Let (V, π, E) and (V', π', E') be two Jacobi vector spaces. A linear map $\varphi : V \to V'$ is said to be a conformal Jacobi map with conformal factor $A := (\alpha, a) \in V^* \times \mathbb{R}^\times$, also called an *A-conformal Jacobi map*, if it is a Jacobi morphism for the conformally equivalent Jacobi structure $\left(\pi^{(A)}, E^{(A)}\right)$ on V. This can be expressed as

$$a\varphi\pi^\sharp\varphi^* = (\pi')^\sharp, \quad \varphi\left(aE + \pi^\sharp\alpha\right) = E'. \tag{2.40}$$

The Jacobi maps are the conformal Jacobi maps with conformal factor $A = (0, 1)$.

At this stage a natural problem, inspired by the differential geometric setting, arises: to associate to a given A-conformal Jacobi map $\varphi : V \to V'$, an appropriate Poisson map $\hat{\varphi}^{(A)}$ from $(V \oplus \mathbb{R}, \hat{\pi})$ to $(V' \oplus \mathbb{R}, \hat{\pi}')$, i.e. to alter the map, not the Jacobi structure. The answer is somehow surprising:

Proposition 2.1.29. *Let (V, π, E) and (V', π', E') be two Jacobi vector spaces. The map $\varphi : V \to V'$ is an A-conformal Jacobi map with $A = (\alpha, a)$, $a > 0$, if and only if the map*

$$\hat{\varphi}^{(A)} : V \oplus \mathbb{R} \to V' \oplus \mathbb{R}, \quad \hat{\varphi}^{(A)}(x \oplus t) := \sqrt{a}\varphi(x) \oplus \left(t - a^{-1}\alpha(x)\right) \tag{2.41}$$

is a Poisson map from $(V \oplus \mathbb{R}, \hat{\pi})$ to $(V' \oplus \mathbb{R}, \hat{\pi}')$.

Proof. The dual of $\hat{\varphi}^{(A)}$ given in (2.41) reads

$$\left(\hat{\varphi}^{(A)}\right)^{*}(\eta' \oplus s') = \sqrt{a}\left(\varphi^{*}\eta' - a^{-1}s'\alpha\right) \oplus s', \tag{2.42}$$

for $\eta' \oplus s' \in (V')^{*} \oplus \mathbb{R}$. Now using (2.25), (2.40), and (2.42), a direct computation gives

$$\hat{\pi}\left(\left(\hat{\varphi}^{(A)}\right)^{*}(\eta'_1 \oplus s'_1), \left(\hat{\varphi}^{(A)}\right)^{*}(\eta'_2 \oplus s'_2)\right) = \hat{\pi}'(\eta'_1 \oplus s'_1, \eta'_2 \oplus s'_2),$$

which shows that the morphism (2.41) is a Poisson map between the Poisson vector spaces $(V \oplus \mathbb{R}, \hat{\pi})$ and $(V' \oplus \mathbb{R}, \hat{\pi}')$.

Conversely, if the map $\hat{\varphi}^{(A)}$ defined in (2.41) is Poisson, i.e. $\hat{\pi'}^{\sharp} = \hat{\varphi}^{(A)}\hat{\pi}^{\sharp}\left(\hat{\varphi}^{(A)}\right)^{*}$, then because of

$$\hat{\varphi}^{(A)}\hat{\pi}^{\sharp}\left(\hat{\varphi}^{(A)}\right)^{*}(\eta \oplus s) = \left(a\varphi\pi^{\sharp}\varphi^{*}(\eta) - s\varphi(aE + \pi^{\sharp}\alpha)\right) \oplus \eta\left(\varphi(aE + \pi^{\sharp}\alpha)\right),$$

the original map, $\varphi : V \to V'$, is an A-conformal Jacobi one. □

Remark 2.1.30. *The unusual square root factor $\sqrt{a}$ in the previous proposition can be avoided via an adapted (to conformal) pairing on the codomain $V' \oplus \mathbb{R}$. Precisely, if one considers the pairing*

$$\langle \bullet, \bullet \rangle : (V' \oplus \mathbb{R}) \times (V'^{*} \oplus \mathbb{R}) \to \mathbb{R}, \quad \langle x' \oplus t', \alpha' \oplus s' \rangle := a(\alpha'(x') + s't'), \tag{2.43}$$

then the linear map φ is conformal if and only if

$$\tilde{\varphi}^{(A)} : V \oplus \mathbb{R} \to V' \oplus \mathbb{R}, \quad \tilde{\varphi}^{(A)}(x \oplus t) := \varphi(x) \oplus \left(t - a^{-1}\alpha(x)\right) \tag{2.44}$$

is Poisson with respect to $\hat{\pi}$ and $\hat{\pi'}$ respectively.

Indeed, if we denote by $(\tilde{\varphi}^{(A)})^{\dagger}$ the dual of linear map (2.44) *with respect to the pairings* (2.20) *and* (2.43)*, it results that*

$$(\tilde{\varphi}^{(A)})^{\dagger}(\eta' \oplus t') = (a\varphi^{*}(\eta') - t'\alpha) \oplus (at').$$

By direct computation based on this dual, one further gets

$$\tilde{\varphi}^{(A)}\hat{\pi}^{\sharp}(\tilde{\varphi}^{(A)})^{\dagger}(\eta' \oplus s') = \left(a\varphi\pi^{\sharp}\varphi^{*}(\eta') - s'\varphi(aE + \pi^{\sharp}\alpha)\right) \oplus \eta'(\varphi(aE + \pi^{\sharp}\alpha)),$$

which proves the claim.

It is noteworthy that the uncommon factor $\sqrt{a}$ is naturally 'renormalized' in the broadest context of L-Jacobi vector spaces [15]. These are the objects of a category, LJVS, with the morphisms L-Jacobi maps. This category is a subcategory in the L-vector spaces category LVS with the morphisms L-maps Let $(L, \mathbb{V})$ and $(L', \mathbb{V}')$ be two L-vector spaces. An L-map is a pair (f, Φ) consisting of a line isomorphism $f : L \to L'$ and a short exact sequence morphism Φ, i.e., the pair of linear maps $\Phi := (\tilde{\varphi}, \varphi)$ that makes the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{R} & \longrightarrow & \hat{V} & \longrightarrow & V & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle \tilde{\varphi}} & & \downarrow{\scriptstyle \varphi} & & \\ 0 & \longrightarrow & \mathbb{R} & \longrightarrow & \hat{V}' & \longrightarrow & V' & \longrightarrow & 0 \end{array}$$

commutative. When the the L-vector spaces are trivial (see Remark 2.1.14) then the line isomorphism reduces to the multiplication with a non-vanishing constant, $f(1) = a$, $a \in \mathbb{R}^\times$, while the liner map $\tilde{\varphi}$ in the short exact sequence morphism reads

$$\tilde{\varphi}(x \oplus t) = \varphi(x) \oplus (\beta(x) + t)$$

with $\beta \in V^*$.

In order to 'isolate' the mentioned subcategory, we invoke the natural L-pairing in L-vector spaces that results from (2.7)

$$\langle \cdot, \cdot \rangle : \tilde{V} \times \hat{V} \to L, \quad \langle \alpha \otimes l, x \rangle := \alpha(x) l,$$

and introduce the L-adjoint map associated with the L-map (f, Φ), $(f, \Phi)^\dagger : \tilde{V}' \to \tilde{V}$, defined by

$$\big\langle (f, \Phi)^\dagger \psi', \delta \big\rangle = f^{-1} \Big\langle \psi, \Phi(\delta) \Big\rangle.$$

With these preparations at hand, we are able to introduce the L-Jacobi maps. Let $(J, L, \mathbb{V})$ and $(J', L', \mathbb{V}')$ be two L-Jacobi vector spaces and (f, Φ) be an L-map. This is said to be an L-Jacobi map if and only if

$$J'^\sharp = \tilde{\varphi} \circ J^\sharp \circ (f, \Phi)^\dagger,$$

where

$$J^\sharp : \tilde{V} \to \tilde{V}^* \otimes L = \hat{V}, \quad J'^\sharp : \tilde{V}' \to \tilde{V}'^* \otimes L' = \hat{V}'.$$

Within this setting, the A-conformal Jacobi map (see Definition 2.1.28) is nothing but a L-Jacobi map (f, Φ) with $f(1) = a$ and

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{R} & \longrightarrow & V \oplus \mathbb{R} & \longrightarrow & V & \longrightarrow & 0\,, \\
 & & \big\downarrow{\scriptstyle 1} & & \big\downarrow{\scriptstyle \tilde{\varphi}} & & \big\downarrow{\scriptstyle \varphi} & & \\
0 & \longrightarrow & \mathbb{R} & \longrightarrow & V' \oplus \mathbb{R} & \longrightarrow & V' & \longrightarrow & 0
\end{array}$$

where $\tilde{\varphi}$ is introduced in (2.44).

2.1.7 Coisotropic subspaces

Coisotropic submanifolds in the Jacobi setting have been studied for instance in [38, 44]. Their main use is in performing Jacobi reduction [61, 57]. In this section we adapt this concept to our linear framework.

Definition 2.1.31. *A linear subspace W of a Jacobi space (V, π, E) is called coisotropic if*

$$\pi^\sharp W^\circ \subseteq W \text{ and } E \in W.$$

The definition is compatible with Poisson coisotropy: a linear subspace W of a Poisson vector space (V, π) is called *coisotropic* if $\pi^\sharp W^\circ \subseteq W$. It also yields the expected definition of a coisotropic subspace of a symplectic vector space (V, ω), namely $W^\omega \subseteq W$, since $\pi^\sharp W^\circ$ coincides with the symplectic orthogonal W^ω.

A coisotropic subspace of a contact space (V, ω_H, E) is, by definition (see [74] for the differential geometric setting), a subspace W that is transversal to H and satisfies the identity

$$(H \cap W)^{\omega_H} \subseteq H \cap W. \tag{2.45}$$

Since $\pi^\sharp W^\circ = (H \cap W)^{\omega_H}$, the Definition 2.1.31 yields in the contact case a special coisotropic subspace W, namely one that satisfies $E \in W$ (which implies that W is transversal to H).

Coisotropic subspaces are the right setting for Jacobi reduction. Given a coisotropic subspace W of a Jacobi vector space (V, π, E), on the quotient space

$$\tilde{W} = W/\pi^\sharp W^\circ$$

one defines a linear Jacobi structure $(\tilde{\pi}, \tilde{E})$, called the reduced Jacobi structure. Here

$$\tilde{E} := E + \pi^\sharp W^\circ$$

and, noticing that its dual is canonically identified with

$$\tilde{W}^* \simeq (\pi^\sharp W^\circ)^\circ / W^\circ, \tag{2.46}$$

one defines the bi-vector

$$\tilde{\pi}(\beta + W^\circ, \gamma + W^\circ) := \pi(\beta, \gamma), \quad \forall \beta, \gamma \in (\pi^\sharp W^\circ)^\circ. \tag{2.47}$$

In the symplectic case we obtain the reduced structure on $\tilde{W} = W/W^\omega$ that comes from the symplectic form ω, namely

$$\tilde{\omega}(u + W^\omega, v + W^\omega) = \omega(u, v), \quad \forall u, v \in W. \tag{2.48}$$

Indeed, we notice that the 1-forms $\beta = i_u\omega$ and $\gamma = i_v\omega$ belong to $(W^\omega)^\circ = (\pi^\sharp W^\circ)^\circ$, hence their classes $\beta + W^\circ = i_{u+W^\omega}\tilde{\omega}$ and $\gamma + W^\circ = i_{v+W^\omega}\tilde{\omega}$ in $\tilde{W}^*$ (see (2.46)) satisfy

$$\tilde{\pi}(\beta + W^\circ, \gamma + W^\circ) \overset{(2.47)}{=} \pi(\beta, \gamma) = \omega(u, v) = \tilde{\omega}(u + W^\omega, v + W^\omega).$$

Thus the reduced symplectic form $\tilde{\omega}$ has the induced Poisson bi-vector $\tilde{\pi}$.

Proposition 2.1.32. *If the Jacobi structure on V is non-degenerate, i.e. it comes from a contact structure (ω_H, E), and the subspace W is coisotropic, then the reduced Jacobi structure $(\tilde{\pi}, \tilde{E})$ on $\tilde{W}$ is non-degenerate too. Moreover, it comes from the contact structure $(\omega_{\tilde{H}}, \tilde{E})$ with contact hyperplane $\tilde{H} = (H \cap W)/\pi^\sharp W^\circ$ and curvature 2-form*

$$\omega_{\tilde{H}}(\pi^\sharp\beta + \pi^\sharp W^\circ, \pi^\sharp\gamma + \pi^\sharp W^\circ) = \omega_H(\pi^\sharp\beta, \pi^\sharp\gamma), \tag{2.49}$$

for all $\pi^\sharp\beta, \pi^\sharp\gamma \in H \cap W$.

Proof. Because $H = \operatorname{Im}\pi^\sharp$, we have the identity $H \cap W = \pi^\sharp((\pi^\sharp W^\circ)^\circ)$, thus $\beta, \gamma \in (\pi^\sharp W^\circ)^\circ$. From (2.47) we get

$$\tilde{\pi}^\sharp(\beta + W^\circ) = \pi^\sharp\beta + \pi^\sharp W^\circ, \quad \forall \beta \in (\pi^\sharp W^\circ)^\circ.$$

Now the relation (2.22) between the curvature 2-form ω_H and the bi-vector π allows a similar reasoning as in the symplectic setting above, which leads to the identity (2.49). □

The reduction in the lcs setting has features of the symplectic reduction [34]. A *coisotropic subspace* W of the lcs vector space (V, ω, λ) has to satisfy, by definition, $W^\omega \subseteq W$ together with one of the equivalent conditions $E \in W$ and $\lambda|_{W^\omega} = 0$ (because $E = \pi^\sharp\lambda$). The lcs reduction (i.e. the Jacobi reduction applied to a coisotropic subspace of an lcs space) yields an lcs space $(\tilde{W}, \tilde{\omega}, \tilde{\lambda})$. Thus, beside the reduced non-degenerate 2-form $\tilde{\omega}$ on $\tilde{W}$ (defined in (2.48)), which is the inverse of $\tilde{\pi}$ (defined in (2.47)), we also get the Lee form $\tilde{\lambda} := \lambda + W^\circ \in \tilde{W}^*$, which corresponds to $\tilde{E} = E + \pi^\sharp W^\circ$ via $\tilde{\pi}$.

The notion of coisotropic subspace is compatible with the Poissonization procedure: $W \subseteq (V, \pi, E)$ is coisotropic if and only if $W \oplus \mathbb{R} \subseteq (V \oplus \mathbb{R}, \hat{\pi})$ is coisotropic. A more general fact (needed later) holds:

Lemma 2.1.33. *Let v_0 be a fixed vector in V. Then the subspace W of $(V, \pi + v_0 \wedge E, E)$ is coisotropic if and only if the subspace*

$$U = \{(x + tv_0, t) : x \in W, t \in \mathbb{R}\}$$

of the Poissonization $(V \oplus \mathbb{R}, \hat{\pi})$ is coisotropic.

Proof. The annihilator of U is

$$U^\circ = \{(\eta, -\eta(v_0)) : \eta \in W^\circ\}.$$

We notice that $\hat{\pi}^\sharp(\eta, -\eta(v_0)) = (\pi^\sharp\eta + \eta(v_0)E, \eta(E)) = ((\pi + v_0 \wedge E)^\sharp\eta + \eta(E)v_0, \eta(E))$. The required equivalence follows. □

Let (V_i, π_i), $i = 1, 2$ be two Poisson vector spaces. The product vector space $V_1 \times V_2$ comes naturally equipped with a product Poisson structure $\pi_1 \times \pi_2$. In the light of the canonical identification $V_1^* \times V_2^* = (V_1 \times V_2)^*$, the product Poisson structure $\pi_1 \times \pi_2$ satisfies

$$(\pi_1 \times \pi_2)^\sharp : V_1^* \times V_2^* \to V_1 \times V_2, \quad (\pi_1 \times \pi_2)^\sharp(\eta_1, \eta_2) = (\pi_1^\sharp\eta_1, \pi_2^\sharp\eta_2). \tag{2.50}$$

Lemma 2.1.34. *Given two Jacobi vector spaces (V_i, π_i, E_i), $i = 1, 2$, the vector space $V_1 \times V_2 \times \mathbb{R}$, endowed with the Jacobi structure*

$$\pi = \pi_1 - \pi_2 + E_1 \wedge 1 + E_2 \wedge 1 \text{ and } E = E_1, \tag{2.51}$$

has the property that its Poissonization is isomorphic to the product $\hat{\pi}_1 \times (-\hat{\pi}_2)$ of the Poissonizations of (π_1, E_1) and $-(\pi_2, E_2)$, through the linear map

$$\psi : (V_1 \times V_2 \times \mathbb{R}) \oplus \mathbb{R} \to (V_1 \oplus \mathbb{R}) \times (V_2 \oplus \mathbb{R}), \quad \psi((x_1, x_2, \tau) \oplus t) = (x_1 \oplus t, x_2 \oplus (t - \tau)). \tag{2.52}$$

Proof. By means of definition (2.52) it results that

$$(\hat{\pi}_1 \times (-\hat{\pi}_2))^\sharp (\eta_1 \oplus s_1, \eta_2 \oplus s_2) = \Big((\pi_1^\sharp\eta_1 - s_1E_1) \oplus \eta_1(E_1), -((\pi_2^\sharp\eta_2 - s_2E_2) \oplus \eta_2(E_2))\Big) \tag{2.53}$$

$$\psi^* (\eta_1 \oplus s_1, \eta_2 \oplus s_2) = (\eta_1, \eta_2, -s_2) \oplus (s_1 + s_2), \tag{2.54}$$

$$\hat{\pi}^\sharp ((\eta_1, \eta_2, \tau) \oplus t) = \Big(\pi_1^\sharp\eta_1 - (t + \tau)E_1, -(\pi_2^\sharp\eta_2 + \tau E_2), \eta_1(E_1) + \eta_2(E_2)\Big) \oplus \eta_1(E_1). \tag{2.55}$$

Putting together the results (2.53)–(2.55), the identity $(\hat{\pi}_1 \times (-\hat{\pi}_2))^\sharp = \psi\hat{\pi}^\sharp\psi^*$ follows. □

A linear map $\varphi : (V_1, \pi_1) \to (V_2, \pi_2)$ is Poisson if and only if its graph is a coisotropic subspace of $V_1 \times V_2$ endowed with the Poisson bi-vector $\pi_1 \times (-\pi_2)$. A similar result holds for conformal Jacobi maps.

Proposition 2.1.35. *Let $\varphi : (V_1, \pi_1, E_1) \to (V_2, \pi_2, E_2)$ be a linear map and $A = (\alpha, a) \in V_1^* \times \mathbb{R}^\times$ a conformal factor with $a > 0$. Then φ is an A-conformal Jacobi map if and only if*

$$W := \{(x, \sqrt{a}\varphi(x), \frac{1}{\sqrt{a}}\alpha(x)) : x \in V_1\}$$

is a coisotropic subspace of the vector space $V_1 \times V_2 \times \mathbb{R}$ with the product Jacobi structure $(\pi, E) = (\pi_1 - \pi_2 + E_1 \wedge 1 + E_2 \wedge 1, E_1)$, introduced in (2.51).

Proof. We use Lemma 2.1.34, Proposition 2.1.29, and Lemma 2.1.33 applied to

$$U = \psi^{-1}(\mathrm{Graph}(\hat{\varphi}^{(A)})) = \{((x, \sqrt{a}\varphi(x), (1 - \sqrt{a})t + \frac{1}{\sqrt{a}}\alpha(x)), t) : x \in V_1, t \in \mathbb{R}\}$$

and to $v_0 = (0, 0, 1 - \sqrt{a}) \in V_1 \times V_2 \times \mathbb{R}$. □

It is noteworthy that, within the more generous category of L-Jacobi vector spaces [15], we can reformulate the previous proposition such that the unpleasant factor $\sqrt{a}$ no longer appears.

Proposition 2.1.36. *Let $\varphi : (V_1, \pi_1, E_1) \to (V_2, \pi_2, E_2)$ be a linear map and $A = (\alpha, a) \in V_1^* \times \mathbb{R}^\times$ a conformal factor. Then φ is an A-conformal Jacobi map if and only if*

$$W := \{(x, \varphi(x), a^{-1}\alpha(x)) : x \in V_1\}$$

is a coisotropic subspace of the vector space $V_1 \times V_2 \times \mathbb{R}$ with the product Jacobi structure $(\pi, E) = (\pi_1 - \pi_2 + E_1 \wedge 1 + E_2 \wedge 1, E_1)$ of (π_1, E_1) and $-(\pi_2, E_2)$, introduced in (2.51).

2.2 Contact dual pairs in purely linear framework

The linear version of a symplectic dual pair consists of a pair of linear Poisson maps (3.97) with symplectic orthogonal kernels [11]. In this section we study the linear version of a contact dual pair, inspired by the Example 3.4 in [73] that describes contact dual pairs in the trivial line bundle case.

2.2.1 Linear contact dual pairs

Definition 2.2.1. Let (V, ω_H, E) be a contact vector space and (V_1, π_1, E_1) and (V_2, π_2, E_2) be two Jacobi vector spaces. The pair of Jacobi maps φ_1, φ_2 with conformal factors $A_1 = (\alpha_1, a_1)$, $A_2 = (\alpha_2, a_2)$,

$$\begin{array}{ccc} & (V, \omega_H, E) & \\ \swarrow^{\varphi_1} & & \searrow^{\varphi_2} \\ (V_1, \pi_1, E_1) & & (V_2, \pi_2, E_2), \end{array} \tag{2.56}$$

forms a *linear contact dual pair* if the following three conditions are satisfied:

1. $X_{A_1} \in \operatorname{Ker}\varphi_2$, $X_{A_2} \in \operatorname{Ker}\varphi_1$,
2. $(H \cap \operatorname{Ker}\varphi_1)^{\omega_H} = H \cap \operatorname{Ker}\varphi_2$,
3. $\pi(\alpha_1, \alpha_2) = \alpha_2(X_{A_1}) - \alpha_1(X_{A_2})$,

with (π, E) representing the non-degenerate Jacobi structure on V associated with the considered contact structure (ω_H, E) (see Example 2.1.10 and Proposition 2.1.18), and X_{A_1}, X_{A_2} denoting the Hamiltonian vectors in the contact vector space V. Since both X_{A_1} and X_{A_2} are transverse to the contact hyperplane H, the point 1 implies the transversality properties

$$H + \operatorname{Ker}\varphi_1 = H + \operatorname{Ker}\varphi_2 = V. \tag{2.57}$$

The importance of point 3 will be revealed in the symplectization procedure from Section 2.2.3, where contact dual pairs are transformed into symplectic dual pairs, as well as in the Section 2.2.4, where the characteristic subspace correspondence in contact dual pairs is described.

The contact dual pair is called *full* if the conformal Jacobi maps φ_1, φ_2 are surjective. In this case

$$\dim V_1 + \dim V_2 = \dim V - 1. \tag{2.58}$$

Indeed, from the transversality (2.57) and point 2 of the definition, we get that $\dim \operatorname{Ker}\varphi_1 + \dim \operatorname{Ker}\varphi_2 = \dim V + 1$. Together with $\dim \operatorname{Ker}\varphi_i = \dim V - \dim V_i$, a consequence of the surjectivity of φ_i, this implies (2.58).

Proposition 2.2.2. *In any linear contact dual pair* (2.56)*, the following identities hold:*

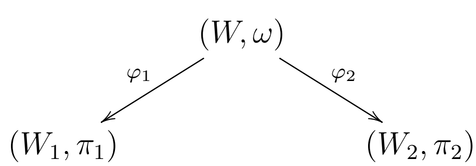
$$H \cap \operatorname{Ker}\varphi_1 = \pi^\sharp(\operatorname{Im}\varphi_2^*), \quad H \cap \operatorname{Ker}\varphi_2 = \pi^\sharp(\operatorname{Im}\varphi_1^*), \tag{2.59}$$

as well as a Howe [58] type property:

$$\pi(\varphi_1^*\eta_1, \varphi_2^*\eta_2) = 0, \quad \forall \eta_1 \in V_1^*, \eta_2 \in V_2^*. \tag{2.60}$$

Proof. The first two identities follow from the Lemma 2.1.12 applied to $\operatorname{Ker}\varphi_i$, $i = 1, 2$. For the Howe type property we notice that $\pi(\varphi_1^*\eta_1, \varphi_2^*\eta_2) = -\eta_1(\varphi_1\pi^\sharp\varphi_2^*\eta_2)$, which vanishes by (2.59). □

Example 2.2.3. Every linear symplectic dual pair can be pulled back to a linear contact dual pair as follows. We start with a symplectic dual pair



The symplectic vector space (W, ω) exhibits the hyperplane $W \leq W \oplus \mathbb{R}$, which is equipped with non-degenerate 2-form $\omega_W = \omega$. By taking the vector $E = 1 \in W \oplus \mathbb{R}$, we display the contact vector space $(W \oplus \mathbb{R}, \omega_W, E = 1)$ associated with the initial symplectic one (W, ω)

The Poisson maps φ_i, for $i = 1, 2$ are Jacobi, and the projection $p : W \oplus \mathbb{R} \to W$ is Jacobi too, thus $\varphi_i \circ p$ are Jacobi maps. We get the linear contact dual pair of Jacobi maps

$$\begin{array}{ccccc} & & (W \oplus \mathbb{R}, \omega, E) & & \\ & \swarrow^{\varphi_1 \circ p} & & \searrow^{\varphi_2 \circ p} & \\ (W_1, \pi_1) & & & & (W_2, \pi_2). \end{array}$$

2.2.2 Contact orthogonality

Definition 2.2.4. *Let U be a subspace of the contact vector space (V, ω_H, E) which is transverse to H, i.e. $H + U = V$. The contact orthogonal of U is defined by*

$$U^{\perp} := (H \cap U)^{\omega_H} \oplus \langle E \rangle. \tag{2.61}$$

The analogous differential geometric notion appears in [50], where it is called pseudo-orthogonal.

Since $\dim(H \cap U) = \dim U - 1$, we get that

$$\dim U + \dim U^{\perp} = \dim V + 1. \tag{2.62}$$

The relation (2.61) can also be written as $U^{\perp} = \pi^{\sharp}(U^{\circ}) \oplus \langle E \rangle$, by Lemma 2.1.12.

Let $\perp_A$ denote the orthogonal w.r.t. $(\omega_H^{(A)}, E^{(A)}) = (a^{-1}\omega_H, X_A)$, the conformally equivalent contact structure on V. This means that

$$U^{\perp_A} := (H \cap U)^{\omega_H} \oplus \langle X_A \rangle, \tag{2.63}$$

because the ω_H-orthogonality and the $\omega_H^{(A)}$-orthogonality coincide.

Proposition 2.2.5. An equivalent set of conditions for the linear contact dual pair (2.56) are

1. The transverse to H subspaces $\operatorname{Ker}\varphi_1$ and $\operatorname{Ker}\varphi_2$ satisfy $(\operatorname{Ker}\varphi_1)^{\perp_{A_1}} = \operatorname{Ker}\varphi_2$ and $(\operatorname{Ker}\varphi_2)^{\perp_{A_2}} = \operatorname{Ker}\varphi_1$,

2. $\pi(\alpha_1, \alpha_2) = \alpha_2(X_{A_1}) - \alpha_1(X_{A_2})$.

Proof. The points 1 and 2 in the Definition 2.2.1 of a linear contact dual pair are equivalent to the following two identities:

$$(\operatorname{Ker}\varphi_1)^{\perp_{A_1}} = \operatorname{Ker}\varphi_2, \quad (\operatorname{Ker}\varphi_2)^{\perp_{A_2}} = \operatorname{Ker}\varphi_1. \tag{2.64}$$

Indeed, $(\operatorname{Ker}\varphi_1)^{\perp_{A_1}} = (H \cap \operatorname{Ker}\varphi_1)^{\omega_H} \oplus \langle X_{A_1} \rangle = (H \cap \operatorname{Ker}\varphi_2) \oplus \langle X_{A_1} \rangle = \operatorname{Ker}\varphi_2$, by (2.63). For the reverse implication, we notice that $H \cap (\operatorname{Ker}\varphi_i)^{\perp_{A_i}} = (H \cap \operatorname{Ker}\varphi_i)^{\omega_H}$ and the Hamiltonian vector $X_{A_i} \in (\operatorname{Ker}\varphi_i)^{\perp_{A_i}}$. □

Example 2.2.6. The simplest method to get linear symplectic dual pairs is with a linear subspace $W \subseteq V$ of the symplectic vector space (V, ω). If W^ω denotes its symplectic orthogonal, then the pair of canonical projections [60]

$$(V/W, \pi_1) \xleftarrow{\;\varphi_1\;} (V, \omega) \xrightarrow{\;\varphi_2\;} (V/W^\omega, \pi_2),$$

where the quotient spaces are endowed with the Poisson structures induced by φ_i, is a full linear symplectic dual pair.

With the help of the contact orthogonal (2.61), we build in a similar way an example of a linear contact dual pair. Let (V, ω_H, E) be a contact vector space and let $A_i = (\alpha_i, a_i) \in V^* \times \mathbb{R}^\times$ be conformal factors that satisfy $\pi(\alpha_1, \alpha_2) = \alpha_2(X_{A_1}) - \alpha_1(X_{A_2})$. Let $W \subseteq V$ be a linear subspace that contains the Hamiltonian vector X_{A_2}. We endow each of the quotient spaces $V_1 = V/W$ and $V_2 = V/W^{\perp_{A_1}}$ with the unique Jacobi structure that makes the canonical projection $\varphi_i : V \to V_i$ a Jacobi map with conformal factor A_i. Then

$$V/W \xleftarrow{\;\varphi_1\;} (V, \omega_H, E) \xrightarrow{\;\varphi_2\;} V/W^{\perp_{A_1}}$$

is a full linear contact dual pair. What remains to be checked is the last identity of point 1 in the Proposition 2.2.5. Since $\operatorname{Ker} \varphi_1 = W$ and $\operatorname{Ker} \varphi_2 = W^{\perp_{A_1}}$, this follows from

$$(W^{\perp_{A_1}})^{\perp_{A_2}} = ((H \cap W)^{\omega_H} \oplus \langle X_{A_1} \rangle)^{\perp_{A_2}} = (H \cap W) \oplus \langle X_{A_2} \rangle = W,$$

where we use the hypothesis $X_{A_2} \in W$.

2.2.3 Symplectization of linear contact dual pairs

As in the differentiable setting, contact dual pairs lead to symplectic dual pairs by a symplectization/Poissonization procedure [73, Section 4.2].

Proposition 2.2.7. *Given a linear contact dual pair*

$$(V_1, \pi_1, E_1) \xleftarrow{\;\varphi_1\;} (V, \omega_H, E) \xrightarrow{\;\varphi_2\;} (V_2, \pi_2, E_2) \tag{2.65}$$

of Jacobi maps with conformal factors $A_1 = (\alpha_1, a_1)$ and $A_2 = (\alpha_2, a_2)$, such that $a_1, a_2 > 0$, the pair of Poisson maps

$$(V_1 \oplus \mathbb{R}, \hat{\pi}_1) \xleftarrow{\;\hat{\varphi}_1^{(A_1)}\;} (V \oplus \mathbb{R}, \hat{\omega}) \xrightarrow{\;\hat{\varphi}_2^{(A_2)}\;} (V_2 \oplus \mathbb{R}, \hat{\pi}_2) \tag{2.66}$$

is a linear symplectic dual pair.

We notice that if the contact dual pair (2.65) is full, then the symplectic dual pair (2.66) is full too.

Proof. By a direct computation based on the definition (2.41) of $\varphi^{(A)}$, and invoking point 1 of the Definition 2.2.1 we get

$$\begin{aligned}\operatorname{Ker}\hat{\varphi}_1^{(A_1)} &= \left\{x \oplus a_1^{-1}\alpha_1(x) : x \in \operatorname{Ker}\varphi_1\right\} \\ &= \left\{x \oplus a_1^{-1}\alpha_1(x) : x \in H \cap \operatorname{Ker}\varphi_1\right\} \oplus \langle X_{A_2} \oplus a_1^{-1}\alpha_1(X_{A_2})\rangle,\end{aligned}$$

and a similar identity for $\operatorname{Ker}\hat{\varphi}_2^{(A_2)}$. In order to prove that (2.66) is a dual pair, first we show that $\hat{\omega}\,(x_1 \oplus t_1, x_2 \oplus t_2) = 0$ for all $x_i \oplus t_i \in \operatorname{Ker}\hat{\varphi}_i^{(A_i)}$, $i = 1, 2$. Due to the above description of the kernel, this follows by checking the identities

$$\begin{aligned}\hat{\omega}\left(x_1 \oplus a_1^{-1}\alpha_1(x_1), x_2 \oplus a_2^{-1}\alpha_2(x_2)\right) = 0,\\ \hat{\omega}\left(x_1 \oplus a_1^{-1}\alpha_1(x_1), X_{A_1} \oplus a_2^{-1}\alpha_2(X_{A_1})\right) = 0,\\ \hat{\omega}\left(X_{A_2} \oplus a_1^{-1}\alpha_1(X_{A_2}), x_2 \oplus a_2^{-1}\alpha_2(x_2)\right) = 0,\\ \hat{\omega}\left(X_{A_2} \oplus a_1^{-1}\alpha_1(X_{A_2}), X_{A_1} \oplus a_2^{-1}\alpha_2(X_{A_1})\right) = 0,\end{aligned}$$

for $x_i \in H \cap \operatorname{Ker}\varphi_i$, $i = 1, 2$. E.g. the last identity uses the point 3 in Definition 2.2.1 of a contact dual pair as follows:

$$\begin{aligned}&\hat{\omega}\left(X_{A_2} \oplus a_1^{-1}\alpha_1(X_{A_2}), X_{A_1} \oplus a_2^{-1}\alpha_2(X_{A_1})\right)\\ &\qquad= \omega\left(X_{A_2}, X_{A_1}\right) + a_2^{-1}\theta(X_{A_2})\alpha_2(X_{A_1}) - a_1^{-1}\theta(X_{A_1})\alpha_1(X_{A_2})\\ &\qquad\qquad= \pi(\alpha_2, \alpha_1) + \alpha_2(X_{A_1}) - \alpha_1(X_{A_2}) = 0.\end{aligned}$$

The vanishing of $\hat{\omega}$ on pairs of vectors that belong to the kernels of the Poisson maps $\hat{\varphi}_1^{(A_1)}$ and $\hat{\varphi}_2^{(A_2)}$ means the inclusion $\operatorname{Ker}\hat{\varphi}_1^{(A_1)} \subseteq \left(\operatorname{Ker}\hat{\varphi}_2^{(A_2)}\right)^{\hat{\omega}}$. Combined with the dimension count

$$\begin{aligned}\dim\operatorname{Ker}\hat{\varphi}_1^{(A_1)} &= \dim\operatorname{Ker}\varphi_1 \overset{(2.64)}{=} \dim\left(\operatorname{Ker}\varphi_2\right)^{\perp_{A_2}} = (\dim V - 1) - (\dim\operatorname{Ker}\varphi_2 - 1) + 1\\ &= \dim\left(V \oplus \mathbb{R}\right) - \dim\operatorname{Ker}\hat{\varphi}_2^{(A_2)} = \dim\left(\operatorname{Ker}\hat{\varphi}_2^{(A_2)}\right)^{\hat{\omega}},\end{aligned}$$

this eventually proves that $\operatorname{Ker}\hat{\varphi}_1^{(A_1)}$ and $\operatorname{Ker}\hat{\varphi}_2^{(A_2)}$ are orthogonal with respect to the symplectic form $\hat{\omega}$, hence (2.66) is a dual pair. □

The condition $a_1, a_2 > 0$ in the statement of Proposition 2.2.7 is not a very restrictive condition because, given Jacobi vector spaces (V, π, E) and (V', π', E'), a linear map $\varphi : V \to V'$, and $A = (\alpha, a) \in V^* \times \mathbb{R}^\times$, then $\varphi : (V, \pi, E) \to (V', \pi', E')$ is an A-conformal Jacobi map if and only if $\varphi : (V, \pi, E) \to (V', -\pi', -E')$ is a conformal Jacobi map with conformal factor $-A = (-\alpha, -a)$. In addition the two Jacobi vector spaces (V', π', E') and $(V', -\pi', -E')$ have the same characteristic subspace.

Proposition 2.2.8. *Given a linear contact dual pair* (2.65) *of Jacobi maps with conformal factors* $A_1 = (\alpha_1, a_1)$ *and* $A_2 = (\alpha_2, a_2)$, *the pair of Poisson maps*

$$\begin{array}{ccc} & (V \oplus \mathbb{R}, \hat{\omega}) & \\ {}^{\tilde{\varphi}_1^{(A_1)}} \swarrow & & \searrow {}^{\tilde{\varphi}_2^{(A_2)}} \\ (V_1 \oplus \mathbb{R}, \hat{\pi}_1) & & (V_2 \oplus \mathbb{R}, \hat{\pi}_2) \end{array} , \tag{2.67}$$

where tilde-type linear maps are defined in (2.44), *is a linear symplectic dual pair.*

Proof. Invoking Remark 2.1.30 and the reasoning flow in the proof of Proposition 2.2.7, the proof of the present claim is immediate from the obvious vector space equality

$$\operatorname{Ker} \tilde{\varphi}^{(A)} = \operatorname{Ker} \hat{\varphi}^{(A)}.$$

□

2.2.4 Characteristic subspace correspondence

We start with the symplectic dual pair setting. We recall that a *symplectic realization* of a Poisson vector space (V', π') is a symplectic vector space (V, ω) together with a linear Poisson map $\varphi : V \to V'$. The characteristic subspace C' of V' is

$$C' = \operatorname{Im} \pi'^{\sharp} = \varphi \pi^{\sharp}(\operatorname{Im} \varphi^*) = \varphi \pi^{\sharp}((\operatorname{Ker} \varphi)^{\circ}) = \varphi\left((\operatorname{Ker} \varphi)^{\omega}\right). \tag{2.68}$$

Thus, the preimage of the characteristic subspace of V' is

$$\varphi^{-1}(C') = \operatorname{Ker} \varphi + (\operatorname{Ker} \varphi)^{\omega}. \tag{2.69}$$

Proposition 2.2.9. *In a full linear symplectic dual pair* (3.97), *the characteristic subspaces* C_1, C_2 *of the two Poisson vector spaces* V_1, V_2 *are related by* $C_2 = \varphi_2(\varphi_1^{-1}(C_1))$. *Their natural symplectic forms* ω_1, ω_2 *satisfy:*

$$\omega = \varphi_1^* \omega_1 + \varphi_2^* \omega_2 \ \textit{on} \ \operatorname{Ker} \varphi_1 + \operatorname{Ker} \varphi_2. \tag{2.70}$$

Proof. The first part of the proposition follows from the orthogonality condition $(\operatorname{Ker} \varphi_1)^{\omega} = \operatorname{Ker} \varphi_2$ together with the identities (2.68) and (2.69). The identity (2.70) is enough to be checked on vectors of the form $Y_i = \pi^{\sharp}(\varphi_i^*(\sigma_i))$ for $\sigma_i \in V_i^*$, $i = 1, 2$, since $\operatorname{Ker} \varphi_1 = (\operatorname{Ker} \varphi_2)^{\omega} = \pi^{\sharp}(\operatorname{Im} \varphi_2^*)$ and similarly for $\operatorname{Ker} \varphi_2$. For instance if $\sigma_1, \bar{\sigma}_1 \in V_1^*$, then

$$\begin{aligned} (\varphi_1^* \omega_1)(\pi^{\sharp} \varphi_1^* \sigma_1, \pi^{\sharp} \varphi_1^* \bar{\sigma}_1) &= \omega_1(\varphi_1 \pi^{\sharp} \varphi_1^* \sigma_1, \varphi_1 \pi^{\sharp} \varphi_1^* \bar{\sigma}_1) = \omega_1(\pi_1^{\sharp} \sigma_1, \pi_1^{\sharp} \bar{\sigma}_1) \\ &= \pi_1(\sigma_1, \bar{\sigma}_1) = \pi(\varphi_1^* \sigma_1, \varphi_1^* \bar{\sigma}_1) = \omega(\pi^{\sharp} \varphi_1^* \sigma_1, \pi^{\sharp} \varphi_1^* \bar{\sigma}_1), \end{aligned}$$

while $(\varphi_2^* \omega_2)(\pi^{\sharp} \varphi_1^* \sigma_1, \pi^{\sharp} \varphi_1^* \bar{\sigma}_1) = 0$. □

The rest of the section is concerned with the analogous results in the setting of linear Jacobi structures and linear contact dual pairs.

Definition 2.2.10. A *contact realization* of a Jacobi vector space V' is a contact vector space V together with a conformal Jacobi linear map $\varphi : V \to V'$.

Lemma 2.2.11. *If $\varphi : (V, \omega_H, E) \to (V', \pi', E')$ is a Jacobi map of factor $A := (\alpha, a) \in V^* \times \mathbb{R}^\times$ with $\operatorname{Ker}\varphi$ transversal to H, then the characteristic subspace of V' is*

$$C' = \varphi\left((\operatorname{Ker}\varphi)^{\perp_A}\right). \tag{2.71}$$

Proof. Similarly to the proof of (2.68), the conclusion follows by a short calculation that uses the definition of the characteristic subspace (2.33) associated with the Jacobi structure (π, E), the definition of the contact orthogonal (2.63), the properties (2.40) of the considered Jacobi map φ, and Lemma 2.1.12. □

If in addition $\alpha \in \operatorname{Im}\varphi^*$, then $(\operatorname{Ker}\varphi)^{\perp_A} = (\operatorname{Ker}\varphi)^{\perp}$.

Corollary 2.2.12. *If $\varphi : (V, \omega_H, E) \to (V', \pi', E')$ is a Jacobi map of factor $A := (\alpha, a) \in V^* \times \mathbb{R}^\times$, then the preimage of the characteristic subspace associated with the Jacobi structure (π', E') reads*

$$\varphi^{-1}(C') = \operatorname{Ker}\varphi + (\operatorname{Ker}\varphi)^{\perp_A}. \tag{2.72}$$

Proposition 2.2.13. *In the full linear contact dual pair (2.56), the characteristic subspaces $C_1 \subseteq V_1$ and $C_2 \subseteq V_2$ correspond to each other in the following sense:*

$$\varphi_1(\varphi_2^{-1}(C_2)) = C_1, \quad \varphi_2(\varphi_1^{-1}(C_1)) = C_2.$$

In addition, the dimensions of C_1 and C_2 have the same parity.

Proof. Let us denote

$$D := \operatorname{Ker}\varphi_1 + \operatorname{Ker}\varphi_2. \tag{2.73}$$

The identities (2.64) and (2.72) lead to $\varphi_1^{-1}(C_1) = D = \varphi_2^{-1}(C_2)$. Moreover, the following computation that uses the surjectivity of φ_i,

$$\operatorname{codim} C_i = \dim V_i - \dim C_i = \dim V - \dim \varphi_i^{-1}(C_i) = \dim V - \dim D,$$

ensures that $C_1 \subseteq V_1$ and $C_2 \subseteq V_2$ have the same codimension. By (2.58) the dimensions of V_1 and V_2 have the same parity, thus the dimensions of C_1 and C_2 have also the same parity. □

Theorem 2.2.14. *In a full linear contact dual pair, the characteristic subspaces of the two Jacobi vector spaces are either both odd dimensional (contact), or both even dimensional (lcs). Moreover,*

1. *If the characteristic subspaces $C_i \subseteq V_i$ are both contact, then their contact structures (θ_i, ω_i) are related to the contact structure (θ, ω) of V by*

$$\theta = a_1\varphi_1^*\theta_1 + a_2\varphi_2^*\theta_2 \text{ on } D \tag{2.74}$$

$$\omega = \alpha_1 \wedge \varphi_1^*\theta_1 + \alpha_2 \wedge \varphi_2^*\theta_2 + a_1\varphi_1^*\omega_1 + a_2\varphi_2^*\omega_2 \text{ on } D \tag{2.75}$$

2. *If the characteristic subspaces $C_i \subseteq V_i$ are both lcs, then their lcs structures (ω_i, λ_i) define a ("connection") 1-form η on D by:*

$$\eta := \varphi_1^*\lambda_1 - a_1^{-1}\alpha_1 = \varphi_2^*\lambda_2 - a_2^{-1}\alpha_2. \tag{2.76}$$

and are related to the contact structure (θ, ω) of V by

$$\omega + \eta \wedge \theta = a_1\varphi_1^*\omega_1 + a_2\varphi_2^*\omega_2 \text{ on } D. \tag{2.77}$$

Proof. Because the subspace $D \subseteq V$ is

$$D = \operatorname{Ker}\varphi_1 + \operatorname{Ker}\varphi_2 \overset{(2.59)}{=} \pi^\sharp(\operatorname{Im}\varphi_1^* + \operatorname{Im}\varphi_2^*) + \langle X_{A_1}\rangle + \langle X_{A_2}\rangle,$$

it is enough to check the identities on vectors of the form: X_{A_i} and $Y_i = \pi^\sharp(\varphi_i^*(\sigma_i))$, $\sigma_i \in V_i^*$, for $i = 1, 2$. One uses the fact that φ_i is an A_i-conformal Jacobi map, thus $\varphi_i(X_{A_i}) = E_i$ and $a_i\varphi_i\pi^\sharp\varphi_i^* = \pi_i^\sharp$, as well as all the three conditions in the contact dual pair definition.

Part 1. If the characteristic subspaces $C_i \subseteq V_i$ are both contact, then their contact structures (θ_i, ω_i) are related to the contact structure (θ, ω) of V by

$$\theta = a_1\varphi_1^*\theta_1 + a_2\varphi_2^*\theta_2 \text{ on } D \tag{2.78}$$

$$\omega = \alpha_1 \wedge \varphi_1^*\theta_1 + \alpha_2 \wedge \varphi_2^*\theta_2 + a_1\varphi_1^*\omega_1 + a_2\varphi_2^*\omega_2 \text{ on } D \tag{2.79}$$

The identity (2.78) holds on X_{A_1}:

$$\theta(X_{A_1}) - a_1\theta_1(\varphi_1(X_{A_1})) - a_2\theta_2(\varphi_2(X_{A_2})) = \theta(a_1E + \pi^\sharp(\alpha_1)) - a_1\theta_1(E_1) - 0 = a_1 - a_1 = 0.$$

The identity (2.79) holds on (X_{A_1}, X_{A_2}) because the left hand side is

$$\omega(X_{A_1}, X_{A_2}) = \omega_H(p_H(X_{A_1}), p_H(X_{A_2})) = \omega_H(\pi^\sharp\alpha_1, \pi^\sharp\alpha_2) = \pi(\alpha_1, \alpha_2),$$

while the right hand side gives the same quantity by the point 3 in the contact dual pair definition:

$$-\alpha_1(X_{A_2})\theta_1(\varphi_1(X_{A_1})) + \alpha_2(X_{A_1})\theta_2(\varphi_2(X_{A_2})) = -a_2\alpha_1(E) + a_1\alpha_2(E) + 2\pi(\alpha_1, \alpha_2) = \pi(\alpha_1, \alpha_2).$$

The same identity (2.79) holds on (Y_1, X_{A_2}) because the left hand side is

$$\omega(Y_1, X_{A_2}) = \omega_H(\pi^\sharp\varphi_1^*\sigma_1, \pi^\sharp\alpha_2) = \pi(\varphi_1^*\sigma_1, \alpha_2),$$

while the only non-zero term in the right hand side is

$$\alpha_2(Y_1)\theta_2(\varphi_2(X_{A_2})) = \alpha_2(\pi^\sharp\varphi_1^*\sigma_1)\theta_2(E_2) = \pi(\varphi_1^*\sigma_1, \alpha_2),$$

after noticing that $\theta_1(\varphi_1(Y_1)) = \pi(\varphi_1^*\sigma_1, \varphi_1^*\theta_1) = a_1^{-1}\pi_1(\sigma_1, \theta_1) = 0$.

Part 2. If the characteristic subspaces $C_i \subseteq V_i$ are both lcs, then their lcs structures (ω_i, λ_i) define a ("connection") 1-form η on D by:

$$\eta := \varphi_1^*\lambda_1 - a_1^{-1}\alpha_1 = \varphi_2^*\lambda_2 - a_2^{-1}\alpha_2. \tag{2.80}$$

and are related to the contact structure (θ, ω) of V by

$$\omega + \eta \wedge \theta = a_1 \varphi_1^* \omega_1 + a_2 \varphi_2^* \omega_2 \text{ on } D. \tag{2.81}$$

The identity (2.80) holds on Y_1 because the right hand side gives

$$\begin{aligned}(\varphi_2^* \lambda_2 - a_2^{-1} \alpha_2)(Y_1) &= \pi(\varphi_1^* \sigma_1, \varphi_2^* \lambda_2) - a_2^{-1} \alpha_2(\pi^\sharp \varphi_1^* \sigma_1) \overset{(2.60)}{=} a_2^{-1}(\varphi_1^* \sigma_1)(\pi^\sharp \alpha_2) \\ &= a_2^{-1}(\varphi_1^* \sigma_1)(X_{A_2} - a_2 E) = -(\varphi_1^* \sigma_1)(E),\end{aligned}$$

while, since $E_1 = \varphi_1(X_{A_1})$ and $E_1 = \pi_1^\sharp(\lambda_1)$, we obtain the same result for the left hand side:

$$\begin{aligned}(\varphi_1^* \lambda_1 - a_1^{-1} \alpha_1)(Y_1) &= \pi(\varphi_1^* \sigma_1, \varphi_1^* \lambda_1) + a_1^{-1}(\varphi_1^* \sigma_1)(\pi^\sharp \alpha_1) \\ &= a_1^{-1} \pi_1(\sigma_1, \lambda_1) + a_1^{-1}(\varphi_1^* \sigma_1)(X_{A_1} - a_1 E) \\ &= -a_1^{-1} \sigma_1(\pi_1^\sharp \lambda_1) + a_1^{-1} \sigma_1(E_1) - (\varphi_1^* \sigma_1)(E) = -(\varphi_1^* \sigma_1)(E).\end{aligned}$$

The same identity (2.80) holds on X_{A_1} because

$$(\varphi_1^* \lambda_1 - a_1^{-1} \alpha_1)(X_{A_1}) = -a_1^{-1} \alpha_1(a_1 E + \pi^\sharp \alpha_1) = \alpha_1(E),$$

while

$$(\varphi_2^* \lambda_2 - a_2^{-1} \alpha_2)(X_{A_1}) = -a_2^{-1} \alpha_2(a_1 E + \pi^\sharp \alpha_1) = -a_2^{-1}(a_1 \alpha_2(E) + \pi(\alpha_1, \alpha_2))$$

and the point 3 in Definition 2.2.1 ensures they coincide.

The identity (2.81) holds on (Y_1, X_{A_2}) because the right hand vanishes by $Y_1 \in \operatorname{Ker} \varphi_2$ and $X_{A_2} \in \operatorname{Ker} \varphi_1$. But

$$\begin{aligned}(\omega + \eta \wedge \theta)(Y_1, X_{A_2}) &= \omega_H(\pi^\sharp \varphi_1^* \sigma_1, \pi^\sharp \alpha_2) + \theta(X_{A_2}) \eta(Y_1) \\ &= \pi(\varphi_1^* \sigma_1, \alpha_2) + a_2(-a_2^{-1} \alpha_2(\pi^\sharp \varphi_1^* \sigma_1)) = 0,\end{aligned}$$

so the left hand side vanishes too. □

2.3 Transversals and dual pairs

We define Poisson transversals in the linear setting. The genuine concept of Poisson transversal was linked to symplectic dual pairs in [27]. Here we extract the properties of linear nature on this subject. In the literature the Poisson transversals can be found also under the name of cosymplectic submanifolds [83] or Poisson submanifolds of the second kind [82]. We have chosen the name Poisson transversals, since it permits an adaptation to the Jacobi setting: Jacobi transversals. These are also called Jacobi submanifolds of the second kind in [24, 54]. The relation between contact dual pairs and Jacobi transversals on their two legs has not been developed in the differential geometric setting. Here we present the linear version, leaving the differential geometric version for a future work.

2.3.1 Poisson transversals and symplectic dual pairs

In this section we are closely following [27].

Definition 2.3.1. A linear subspace X of a Poisson vector space (V, π) is called a *Poisson transversal* if it satisfies

$$V = X \oplus \pi^\sharp X^\circ. \tag{2.82}$$

Recall that the characteristic subspace $C = \operatorname{Im} \pi^\sharp$ is endowed with the linear symplectic form ω_C induced by π as in (2.30). For any subspace X of V, we have

$$\operatorname{Ker}(\omega_C|_{X \cap C}) = X \cap \pi^\sharp X^\circ. \tag{2.83}$$

Indeed, we know that this kernel is equal to $X \cap C \cap \pi^\sharp (X \cap C)^\circ$, hence (2.83) holds because of the vector subspace identities $C^\circ = \operatorname{Ker} \pi^\sharp$ and $(X \cap C)^\circ = X^\circ + C^\circ$.

Proposition 2.3.2. *X is a Poisson transversal if and only if it satisfies two conditions:*

1. *X is transverse to C,*
2. *$X \cap C$ is a symplectic subspace of the symplectic space (C, ω_C).*

Proof. Using (2.83), we obtain that point 2 is equivalent to $X \cap \pi^\sharp X^\circ = 0$. On the other hand point 1 is equivalent to $\dim X + \dim \pi^\sharp X^\circ = \dim V$, which concludes the characterization of Poisson transversals by 1 and 2. □

Remark 2.3.3 (Induced Poisson structure)**.** Each Poisson transversal X of the Poisson vector space (V, π) inherits a canonical Poisson structure π_X given by

$$\pi_X^\sharp := p_X \circ \pi^\sharp \circ p_X^*, \tag{2.84}$$

where $p_X : V \to X$ denotes the projection on the first factor in the decomposition (2.82). An equivalent definition of π_X is: the unique Poisson structure on X such that the projection $p_X : (V, \pi) \to (X, \pi_X)$ is a Poisson map.

Moreover, the characteristic subspace $C_X := \operatorname{Im} \pi_X^\sharp$ of X of the Poisson vector space X is the projection of the characteristic subspace of V, i.e. $C_X = p_X(C)$.

As a consequence, if the original Poisson structure on V is transitive (i.e. non-degenerate), then the induced one on X is transitive (i.e. non-degenerate) too.

Corollary 2.3.4. *The Poisson transversals of a symplectic vector space are its symplectic subspaces.*

Example 2.3.5. Any linear complement of the characteristic subspace $C \subseteq V$, is a Poisson transversal. Its induced Poisson structure is trivial.

Lemma 2.3.6. *Let $\varphi : (V, \pi) \to (V_1, \pi_1)$ be a Poisson map and let the linear subspace $X_1 \subseteq V_1$ be a Poisson transversal. Then the following hold true:*

1. *The map φ is transverse to X_1;*

2. *The subspace* $X := \varphi^{-1}(X_1)$ *of* V *is a Poisson transversal;*

3. $\varphi\left(\pi^\sharp X^\circ\right) = \pi_1^\sharp X_1^\circ$*;*

4. *The restriction* $\varphi_X : X \to X_1$ *is a Poisson map with respect to the induced Poisson structures* π_X *and* $(\pi_1)_{X_1}$*.*

Proof. By the Poisson map property of φ written as $\varphi\pi^\sharp\varphi^* = \pi_1^\sharp$ we have that $\pi_1^\sharp X_1^\circ \subseteq \operatorname{Im}\varphi$. Hence $V_1 = X_1 \oplus \pi_1^\sharp X_1^\circ = X_1 + \operatorname{Im}\varphi$, which shows 1.

The image of an arbitrary element $v \in V$ by φ has a unique decomposition as

$$\varphi(v) = x_1 + \pi_1^\sharp \eta_1, \quad x_1 \in X_1,\ \eta_1 \in X_1^\circ.$$

The element $x := v - \pi^\sharp\varphi^*\eta_1$ satisfies $\varphi(x) = x_1$, hence $x \in X$. Since $\varphi^*\eta_1 \in \varphi^* X_1^\circ \subseteq X^\circ$, we get $v = x + \pi^\sharp\varphi^*\eta_1 \in X + \pi^\sharp X^\circ$, hence the decomposition $V = X + \pi^\sharp X^\circ$, a direct sum decomposition by a dimension count. We conclude that $X \subseteq V$ is a Poisson transversal.

The inclusion $\pi_1^\sharp X_1^\circ \subseteq \varphi(\pi^\sharp X^\circ)$ follows from the relations $\pi_1^\sharp = \varphi\circ\pi^\sharp\circ\varphi^*$ and $\varphi^* X_1^\circ \subseteq X^\circ$. For the converse inclusion, let $\beta \in X^\circ$. We find $x_1 \in X_1$ and $\eta_1 \in X_1^\circ$ such that $\varphi\left(\pi^\sharp\beta\right) = x_1 + \pi_1^\sharp\eta_1$. Since $x_1 = \varphi\left(\pi^\sharp\left(\beta - \varphi^*\eta_1\right)\right) \in X_1$, the element $\pi^\sharp\left(\beta - \varphi^*\eta_1\right) \in X \cap \pi^\sharp X^\circ$ is zero, so that $\varphi\left(\pi^\sharp\beta\right) = \pi_1^\sharp\eta_1$. Thus $\varphi\left(\pi^\sharp X^\circ\right) \subseteq \pi_1^\sharp X_1^\circ$ and the identity 3 follows.

Let us denote by p_X and p_{X_1} the projections on the first factor associated with the decompositions $V = X \oplus \pi^\sharp X^\circ$ and $V_1 = X_1 \oplus \pi_1^\sharp X_1^\circ$ respectively. Because $\varphi(\pi^\sharp X^\circ) = \pi_1^\sharp X_1^\circ$ (by point 3), there exists a unique linear map

$$\varphi_X : X \to X_1 \text{ such that } p_{X_1} \circ \varphi = \varphi_X \circ p_X. \tag{2.85}$$

The short calculation

$$\varphi_X\pi_X^\sharp\varphi_X^* \overset{(2.84)}{=} \varphi_X p_X \pi^\sharp p_X^* \varphi_X^* \overset{(2.85)}{=} p_{X_1}\varphi\pi^\sharp\varphi^* p_{X_1}^* = p_{X_1}\pi_1^\sharp p_{X_1}^* = \pi_{X_1}^\sharp$$

ensures that φ_X is a Poisson map. □

The next proposition answers the question: when is a pair of Poisson maps again a Poisson map?

Proposition 2.3.7. *Given two linear maps* φ_1 *and* φ_2 *defined on the same symplectic vector space* (V, ω)*, the linear map*

$$\varphi := (\varphi_1, \varphi_2) : V \to (V_1 \times V_2, \pi_1 \times \pi_2) \tag{2.86}$$

is Poisson if and only if $(\operatorname{Ker}\varphi_1)^\omega \subseteq \operatorname{Ker}\varphi_2$ *and both* φ_1 *and* φ_2 *are Poisson.*

Proof. Let π denote the linear Poisson structure on the symplectic space, i.e. $\pi^\sharp = (\omega^\flat)^{-1}$. The condition that the map φ is Poisson is written as $\varphi\pi^\sharp\varphi^* = (\pi_1 \times \pi_2)^\sharp$. The conclusion follows by a direct computation involving the expression (2.50) of $(\pi_1 \times \pi_2)^\sharp$, and that of the dual of φ, namely $\varphi^*(\eta_1, \eta_2) = \varphi_1^*(\eta_1) + \varphi_2^*(\eta_2)$. Indeed, for all $\eta_i \in V_i^*$, $i = 1, 2$, we get

$$\varphi\pi^\sharp\varphi^*(\eta_1, \eta_2) = (\varphi_1\pi^\sharp(\varphi_1^*(\eta_1) + \varphi_2^*(\eta_2)), \varphi_2\pi^\sharp(\varphi_1^*(\eta_1) + \varphi_2^*(\eta_2)))$$

$$= (\varphi_1 \pi^\sharp \varphi_1^* \eta_1, \varphi_2 \pi^\sharp \varphi_2^* \eta_2) + (\varphi_1 \pi^\sharp \varphi_2^* \eta_2, \varphi_2 \pi^\sharp \varphi_1^* \eta_1)$$
$$(\pi_1 \times \pi_2)^\sharp(\eta_1, \eta_2) = (\pi_1^\sharp \eta_1, \pi_2^\sharp \eta_2).$$

This means that the map φ is Poisson if and only if both maps φ_1, φ_2 are Poisson, and the identities $\varphi_1 \pi^\sharp \varphi_2^* = 0$ and $\varphi_2 \pi^\sharp \varphi_1^* = 0$ hold. The first identity can be written as $(\operatorname{Ker} \varphi_2)^\omega = \pi^\sharp(\operatorname{Ker} \varphi_2)^\circ = \pi^\sharp \operatorname{Im} \varphi_2^* \subseteq \operatorname{Ker} \varphi_1$ and the second one as the inclusion $(\operatorname{Ker} \varphi_1)^\omega \subseteq \operatorname{Ker} \varphi_2$, which is equivalent to the first inclusion. □

Theorem 2.3.8. *We consider a linear symplectic dual pair*

$$\begin{array}{ccc} & (V,\omega) & \\ {}^{\varphi_1}\swarrow & & \searrow^{\varphi_2} \\ (V_1,\pi_1) & & (V_2,\pi_2) \end{array}$$

and linear Poisson transversals $X_i \subseteq V_i$, $i = 1, 2$. Then the vector subspace $W := \varphi_1^{-1}(X_1) \cap \varphi_2^{-1}(X_2)$ of (V, ω) is symplectic (with induced symplectic form denoted by ω_W) and the pair of Poisson maps obtained by restriction of φ_i, $i = 1, 2$,

$$\begin{array}{ccc} & (W,\omega_W) & \\ {}^{(\varphi_1)_W}\swarrow & & \searrow^{(\varphi_2)_W} \\ (X_1,\pi_{X_1}) & & (X_2,\pi_{X_2}) \end{array}, \tag{2.87}$$

form a linear symplectic dual pair.

Proof. Knowing that X_1 and X_2 are two Poisson transversals in (V_1, π_1) and (V_2, π_2) respectively, the product $X_1 \times X_2$ is a Poisson transversal in $(V_1 \times V_2, \pi_1 \times \pi_2)$, because of the identity $(\pi_1 \times \pi_2)^\sharp (X_1 \times X_2)^\circ = \pi_1^\sharp X_1^\circ \times \pi_2^\sharp X_2^\circ$.

Using the inclusion $(\operatorname{Ker} \varphi_1)^\omega \subseteq \operatorname{Ker} \varphi_2$ (that comes from the symplectic dual pair condition $(\operatorname{Ker} \varphi_1)^\omega = \operatorname{Ker} \varphi_2$) and applying the Proposition 2.3.7, we get that $\varphi = (\varphi_1, \varphi_2)$ is a Poisson map. Invoking Lemma 2.3.6, we get that $W := \varphi^{-1}(X_1 \times X_2) = \varphi_1^{-1}(X_1) \cap \varphi_2^{-1}(X_2)$ is a Poisson transversal in (V, ω), hence a symplectic subspace.

Again by Lemma 2.3.6, the restriction

$$\varphi_W : W \to X_1 \times X_2, \quad \varphi_W := ((\varphi_1)_W, (\varphi_2)_W),$$

of φ is a Poisson map. Now using the Proposition 2.3.7, it follows on one hand that the restrictions $(\varphi_i)_W$, $i = 1, 2$, are Poisson maps (that completes point 2 in the theorem), and on the other hand that

$$(\operatorname{Ker}(\varphi_1)_W)^{\omega_W} \subseteq \operatorname{Ker}(\varphi_2)_W. \tag{2.88}$$

The symplectic form ω vanishes on $\operatorname{Ker} \varphi_1 \times \operatorname{Ker} \varphi_2$, hence ω_W vanishes as well on $\operatorname{Ker}(\varphi_1)_W \times \operatorname{Ker}(\varphi_2)_W$. The reverse inclusion to (2.88) follows, showing that (2.87) is a dual pair. □

The dual pair (2.87) is full if and only if the original one is also full.

2.3.2 Jacobi transversals and contact dual pairs

In this section we develop the linear algebraic theory of Jacobi transversals, which will be later used for obtaining new contact dual pairs from a given one. The differential geometric version of these results will be treated elsewhere.

Definition 2.3.9. A linear subspace X of the Jacobi vector space (V, π, E) is called a *linear Jacobi transversal* if

$$V = X \oplus \pi^\sharp X^\circ. \tag{2.89}$$

Notice that being a Jacobi transversal depends only on the bi-vector π and not on the Reeb vector E.

There is a characterization of Jacobi transversals, like that of Poisson transversals in Proposition 2.3.2, that uses the characteristic subspace $C = \operatorname{Im}\pi^\sharp + \langle E\rangle$ and the non-degenerate 2-form ω_C on $\operatorname{Im}\pi^\sharp$ given by (2.30).

Proposition 2.3.10. *The subspace X is a Jacobi transversal of (V, π, E) if and only if it satisfies the following two conditions:*

1. *X is transverse to $\operatorname{Im}\pi^\sharp$,*

2. *ω_C restricted to $X \cap \operatorname{Im}\pi^\sharp$ is nondegenerate.*

In the case $E \in \operatorname{Im}\pi^\sharp$, the characteristic subspace $C = \operatorname{Im}\pi^\sharp$ is even dimensional and it carries an lcs structure (ω_C, E) (see the first item in Proposition 2.1.21). For any Jacobi transversal X, the subspace $X \cap C$ is an lcs subspace of C.

In the case $E \notin \operatorname{Im}\pi^\sharp$, the characteristic subspace $C = \operatorname{Im}\pi^\sharp \oplus \langle E\rangle$ is odd dimensional and it carries a contact structure $(C, \omega_H = \omega_C, E)$ (see the second item in Proposition 2.1.21). For any Jacobi transversal X, the subspace $X \cap C$ is a contact subspace of C.

Proof. This proof relies on the identity (2.83), proven for the Poisson setting, which holds in this Jacobi setting too. Assume first that X is a Poisson transversal, so it satisfies (2.89). Then $V = X + \operatorname{Im}\pi^\sharp$, so point 1 holds. Point 2 is equivalent to $X \cap \pi^\sharp X^\circ = 0$ by (2.83), so it holds too.

Conversely, we assume that X satisfies 1 and 2. By taking into account that $V = X + \operatorname{Im}\pi^\sharp$, we get that $0 = X^\circ \cap (\operatorname{Im}\pi^\sharp)^\circ = X^\circ \cap \operatorname{Ker}\pi^\sharp$. Thus $\pi^\sharp$ is injective on X°, hence

$$\dim \pi^\sharp X^\circ = \dim X^\circ = \dim V - \dim X. \tag{2.90}$$

Using the point 2, we get by (2.83) that $X \cap \pi^\sharp X^\circ = 0$, which, with the dimension count (2.90), proves the direct sum decomposition (2.89). □

Remark 2.3.11 (Induced Jacobi structure)**.** Each Jacobi transversal X of the Jacobi vector space (V, π, E) inherits a canonical Jacobi structure (π_X, E_X),

$$\pi_X^\sharp := p_X \circ \pi^\sharp \circ p_X^*, \quad E_X := p_X(E), \tag{2.91}$$

with $p_X : V \to X$ denoting the projection on the first factor in (2.89). This is the unique Jacobi structure on X such that the projection p_X is a Jacobi map.

In particular p_X is a Jacobi map that satisfies $\operatorname{Im}\pi^\sharp_X = p_X\left(\operatorname{Im}\pi^\sharp\right)$, thus the characteristic subspace $C_X := \operatorname{Im}\pi^\sharp_X + \langle E_X\rangle$ of X is the image of the characteristic subspace $C \subseteq V$, i.e. $C_X = p_X(C)$. Moreover, $E \in \operatorname{Im}\pi^\sharp$ if and only if $E_X \in \operatorname{Im}\pi^\sharp_X$. The direct implication is obvious. So we give an argument for the reverse implication. Assuming that $E_X \in \operatorname{Im}\pi^\sharp$, there exists $\eta \in V^*$ such that $E_X = \pi^\sharp_X i^*_X\eta$. By (2.91) we get $p_X(E) = p_X(\pi^\sharp p^*_X(i^*_X\eta))$, so that $E - \pi^\sharp p^*_X(i^*_X\eta) \in \operatorname{Ker} p_X = \pi^\sharp X^\circ$, which implies $E \in \operatorname{Im}\pi^\sharp$.

As a consequence, the original Jacobi structure (π, E) is transitive if and only if the induced one, (π_X, E_X), is transitive. They also have the same type: lcs or contact.

Corollary 2.3.12. *The Jacobi transversals of a contact vector spaces are its contact vector subspaces, and the Jacobi transversals of an lcs vector space are its lcs subspaces.*

Example 2.3.13. Any linear complement X of $\operatorname{Im}\pi^\sharp$ in V is a Jacobi transversal. Moreover, the induced Jacobi structure (π_X, E_X) has trivial bi-vector $\pi_X = 0$. The Reeb vector E_X also vanishes if and only if $E \in \operatorname{Im}\pi^\sharp$.

Proposition 2.3.14. *Let X be a vector subspace of the Jacobi vector space (V, π, E), with $E \in X$. Then X is a Jacobi transversal of V if and only if $X \oplus \mathbb{R}$ is a Poisson transversal of its Poissonization $(V \oplus \mathbb{R}, \hat{\pi})$ (defined in (2.25)).*

Proof. The annihilator of $X \oplus \mathbb{R}$ in $V \oplus \mathbb{R}$ coincides with the annihilator X° of X in V. By (2.26) we have that $\hat{\pi}^\sharp(\eta \oplus s) = (\pi^\sharp(\eta) - sE) \oplus \eta(E)$, hence the image of the annihilator by $\hat{\pi}^\sharp$ is

$$\hat{\pi}^\sharp((X \oplus \mathbb{R})^\circ) = \{\pi^\sharp(\eta) \oplus \eta(E) : \eta \in X^\circ\} = \pi^\sharp(X^\circ) \oplus 0,$$

because $X^\circ \subseteq E^\circ$. In the light of this identity, the requested equivalence follows. □

Lemma 2.3.15. *Let $\varphi : V \to V_1$ be an A-conformal Jacobi map, $A = (\alpha, a) \in V^* \times \mathbb{R}^\times$, and let X_1 be a Jacobi transversal of (V_1, π_1, E_1). Then the following hold true:*

1. *The map φ is transverse to X_1;*
2. *The subspace $X := \varphi^{-1}(X_1)$ is a Jacobi transversal of (V, π, E);*
3. $\varphi\left(\pi^\sharp X^\circ\right) = \pi_1^\sharp X_1^\circ$;
4. *The restriction $\varphi_X : X \to X_1$ is a conformal Jacobi map with respect to the induced Jacobi structures, for the conformal factor $A_X = (\alpha|_X, a) \in X^* \times \mathbb{R}^\times$.*

Proof. The fact that φ is an A-conformal Jacobi map translates into $a\varphi\pi^\sharp\varphi^* = \pi_1^\sharp$ and $\varphi\left(E^{(A)}\right) = E_1$, thus we have $V_1 = X_1 \oplus \pi_1^\sharp X_1^\circ \subseteq X_1 + \operatorname{Im}\varphi = \operatorname{Im}\varphi + X_1$, which shows item 1. The items 2 and 3 have the same proof as their counterparts in Lemma 2.3.6. To prove item 4, we first check that $a\varphi_X\pi^\sharp_X\varphi^*_X = \pi^\sharp_{X_1}$:

$$a\varphi_X\pi^\sharp_X\varphi^*_X \overset{(2.91)}{=} a\varphi_X p_X\pi^\sharp p^*_X\varphi^*_X \overset{(2.85)}{=} a p_{X_1}\varphi\pi^\sharp\varphi^* p^*_{X_1} = p_{X_1}\pi_1^\sharp p^*_{X_1} \overset{(2.91)}{=} \pi^\sharp_{X_1}.$$

We notice that the conformal factor A_X can be written as $(i^*_X\alpha, a)$, with $i_X : X \to V$ the inclusion. To verify the equality $\varphi_X(E^{(A_X)}) = E_{X_1}$, we compute

$$\varphi_X(E^{(A_X)}) = \varphi_X(aE_X + \pi^\sharp_X i^*_X\alpha) \overset{(2.91)}{=} \varphi_X p_X(aE + \pi^\sharp p^*_X i^*_X\alpha) \overset{(2.85)}{=} a p_{X_1}\varphi(E) + p_{X_1}\varphi\pi^\sharp(i_X p_X)^*\alpha$$

$$= ap_{X_1}\varphi(E) + p_{X_1}\varphi\pi^\sharp(\alpha) = p_{X_1}\varphi(E^{(A)}) = p_{X_1}(E_1) = E_{X_1},$$

using at step four the identity $\varphi\left(\pi^\sharp X^\circ\right) = \operatorname{Ker} p_{X_1}$ (that follows from item 3) applied to $\alpha - (i_X p_X)^*\alpha \in X^\circ$. Thus φ_X is an A_X-conformal Jacobi map. □

When does a pair of conformal Jacobi maps build a conformal Jacobi map? It turns out that the Jacobi structure on the product of Jacobi vector spaces has to depend on the two conformal factors. Moreover, the kernels of the two maps have to satisfy an orthogonality condition.

Proposition 2.3.16. *We consider two linear maps $\varphi_i : V \to V_i$, $i = 1, 2$, defined on the contact vector space (V, ω_H, E), and two pairs $A_i = (\alpha_i, a_i) \in V^* \times \mathbb{R}^\times$, $i = 1, 2$, such that*

$$X_{A_1} \in \operatorname{Ker}\varphi_2 \text{ and } X_{A_2} \in \operatorname{Ker}\varphi_1. \tag{2.92}$$

Let $V_1 \times V_2$ be endowed with the product Jacobi structure

$$\bar{\pi} := (a_2\pi_1) \times (a_1\pi_2), \quad \bar{E} := (\tfrac{1}{2}a_2E_1, \tfrac{1}{2}a_1E_2). \tag{2.93}$$

Then the linear map $\varphi := (\varphi_1, \varphi_2) : V \to V_1 \times V_2$ is conformal Jacobi of factor

$$A = (\alpha, a) := (\tfrac{1}{2}a_2\alpha_1 + \tfrac{1}{2}a_1\alpha_2, a_1a_2) \in V^* \times \mathbb{R}^\times \tag{2.94}$$

if and only if the identity

$$(\operatorname{Ker}\varphi_1 \cap H)^{\omega_H} \subseteq \operatorname{Ker}\varphi_2 \cap H$$

holds true and φ_i are conformal Jacobi maps of factors A_i, $i = 1, 2$.

Proof. The map φ is A-conformal Jacobi if and only if the identities

$$\varphi(X_A) = \bar{E}, \quad a\varphi\pi^\sharp\varphi^* = \bar{\pi}^\sharp \tag{2.95}$$

hold true, with (π, E) the induced Jacobi structure on the contact vector space V. The first identity in (2.95) is equivalent to

$$\varphi_1(X_{A_1}) = E_1 \text{ and } \varphi_2(X_{A_2}) = E_2. \tag{2.96}$$

This follows from the computation:

$$\begin{aligned}\varphi(X_A) &= \varphi\left(a_1a_2E + \tfrac{1}{2}a_2\pi^\sharp\alpha_1 + \tfrac{1}{2}a_1\pi^\sharp\alpha_2\right) = (\varphi_1, \varphi_2)\left(\tfrac{1}{2}a_1X_{A_2} + \tfrac{1}{2}a_2X_{A_1}\right) \\ &\overset{(2.92)}{=} \left(\tfrac{1}{2}a_2\varphi_1(X_{A_1}), \tfrac{1}{2}a_1\varphi_2(X_{A_2})\right).\end{aligned}$$

The second identity in (2.95) is equivalent to the following four identities:

$$\pi_1^\sharp = a_1\varphi_1\pi^\sharp\varphi_1^*, \quad \pi_2^\sharp = a_2\varphi_2\pi^\sharp\varphi_2^* \tag{2.97}$$

$$\varphi_1\pi^\sharp\varphi_2^* = 0, \quad \varphi_2\pi^\sharp\varphi_1^* = 0. \tag{2.98}$$

This follows from the computation for $\eta_i \in V_i^*$, $i = 1, 2$:

$$a\varphi\pi^\sharp\varphi^*(\eta_1, \eta_2) = a_1a_2(\varphi_1, \varphi_2)\pi^\sharp(\varphi_1^*\eta_1 + \varphi_2^*\eta_2)$$

$$= a_1a_2(\varphi_1\pi^\sharp\varphi_1^*\eta_1 + \varphi_1\pi^\sharp\varphi_2^*\eta_2, \varphi_2\pi^\sharp\varphi_1^*\eta_1 + \varphi_2\pi^\sharp\varphi_2^*\eta_2)$$

and from the expression $\bar{\pi}^\sharp(\eta_1, \eta_2) = (a_2\pi_1^\sharp\eta_1, a_1\pi_2^\sharp\eta_2)$.

Using (2.24), the first identity in (2.98) can be rewritten as $(H\cap\operatorname{Ker}\varphi_2)^{\omega_H} = \pi^\sharp(\operatorname{Ker}\varphi_2)^\circ = \pi^\sharp \operatorname{Im}\varphi_2^* \subseteq H \cap \operatorname{Ker}\varphi_1$ and the second one as the inclusion $(H \cap \operatorname{Ker}\varphi_1)^{\omega_H} \subseteq H \cap \operatorname{Ker}\varphi_2$, which is equivalent with the first inclusion. The identities (2.96) and (2.97) are equivalent to the fact that φ_i is A_i-conformal Jacobi map for $i = 1, 2$. □

Theorem 2.3.17. *We consider a linear contact dual pair*

$$\begin{array}{ccc} & (V, \omega_H, E) & \\ {\scriptstyle \varphi_1}\swarrow & & \searrow{\scriptstyle \varphi_2} \\ (V_1, \pi_1, E_1) & & (V_2, \pi_2, E_2) \end{array},$$

with conformal factors $A_i = (\alpha_i, a_i) \in V^* \times \mathbb{R}^\times$, and two linear Jacobi transversals $X_i \subseteq V_i$, $i = 1, 2$. Then the vector subspace

$$W := \varphi_1^{-1}(X_1) \cap \varphi_2^{-1}(X_2), \tag{2.99}$$

is a contact subspace of (V, ω_H, E), and the pair of conformal Jacobi maps with conformal factors $(A_i)_W := (\alpha_i|_W, a_i)$, obtained by restriction of φ_i, $i = 1, 2$,

$$\begin{array}{ccc} & W & \\ {\scriptstyle (\varphi_1)_W}\swarrow & & \searrow{\scriptstyle (\varphi_2)_W} \\ (X_1, \pi_{X_1}, E_1) & & (X_2, \pi_{X_2}, E_2) \end{array}, \tag{2.100}$$

form a linear contact dual pair.

Proof. The subspace $X_1\times X_2$ is a Jacobi transversal of $\left(V_1 \times V_2, \bar{\pi}, \bar{E}\right)$. Because of the contact dual pair conditions (the first two axioms in Definition 2.2.1) we can apply Proposition 2.3.16, so that $\varphi = (\varphi_1, \varphi_2)$ is a conformal Jacobi map with conformal factor A given in (2.94). Thus, the subspace W in (2.99), which can be expressed as $W = \varphi^{-1}(X_1 \times X_2)$, is a Jacobi transversal of the contact vector space (V, ω_H, E), by Lemma 2.3.15, hence a contact subspace, by Corollary 2.3.12. The induced contact structure on W is given by the hyperplane $H_W = H \cap W$, the curvature form $\omega_{H_W} = \omega_H|_{H\cap W}$ and the Reeb vector $E_W := p_W(E)$, where $p_W : V \to W$ denotes the projection on the first summand for $V = W \oplus \pi^\sharp W^\circ$.

Again by Lemma 2.3.15, the restriction

$$\varphi_W : W \to X_1 \times X_2, \quad \varphi_W := ((\varphi_1)_W, (\varphi_2)_W), \tag{2.101}$$

of φ is a Jacobi map with conformal factor

$$A_W = (\alpha|_W, a) = \left(\tfrac{1}{2}a_2\alpha_1|_W + \tfrac{1}{2}a_1\alpha_2|_W, a_1a_2\right).$$

The map φ_W is characterized by $\varphi_W \circ p_W = (p_{X_1\times X_2}) \circ \varphi$, with $p_{X_1\times X_2}$ the natural projection associated to the Jacobi transversal $X_1 \times X_2$. Thus, its two components satisfy the identity

$$(\varphi_i)_W \circ p_W = p_{X_i} \circ \varphi_i, \quad i = 1, 2, \tag{2.102}$$

which further leads to $p_W\left(E_{A_1}\right) \in \operatorname{Ker}(\varphi_2)_W$ and $p_W\left(E_{A_2}\right) \in \operatorname{Ker}(\varphi_1)_W$. In addition,

$$p_W\left(E_{A_i}\right) = a_i p_W(E) + p_W\pi^\sharp(\alpha_i - (i_W p_W)^*\alpha_i) + (p_W\pi^\sharp p_W^*) i_W^*\alpha_i = a_i E_W + \pi_W^\sharp i_W^*\alpha_i = X_{(A_i)_W},$$

due to the fact that

$$\alpha - (i_W p_W)^*\,\alpha \in W^\circ, \quad \alpha \in V^*.$$

We conclude that

$$X_{(A_1)_W} \in \operatorname{Ker}(\varphi_2)_W \quad \text{and} \quad X_{(A_2)_W} \in \operatorname{Ker}(\varphi_1)_W,$$

which permits to apply Proposition 2.3.16 to the conformal Jacobi map (2.101). On one hand we obtain that the maps $(\varphi_i)_W$, $i = 1, 2$ are conformal Jacobi of factors $(A_i)_W := (i_W^*\alpha_i, a_i)$ and on the other hand we get the inclusion

$$(H_W \cap \operatorname{Ker}(\varphi_1)_W)^{\omega_{H_W}} \subseteq H_W \cap \operatorname{Ker}(\varphi_2)_W. \tag{2.103}$$

But the curvature form ω_H vanishes on $(H \cap \operatorname{Ker}\varphi_1) \times (H \cap \operatorname{Ker}\varphi_2)$, hence ω_{H_W} vanishes as well on $(H_W \cap \operatorname{Ker}(\varphi_1)_W) \times (H_W \cap \operatorname{Ker}(\varphi_2)_W)$ and the reverse inclusion to (2.103) follows, showing that

$$(\operatorname{Ker}(\varphi_1)_W \cap H_W)^{\omega_{H_W}} = \operatorname{Ker}(\varphi_2)_W \cap H_W.$$

The contact dual pair conditions 1 and 2 in Definition 2.2.1 have been checked above, and condition 3 follows immediately from the one for the given dual pair, thus the pair (2.100) forms a linear contact dual pair. □

The contact dual pair (2.100) is full if and only if the original one is full too.

Chapter 3

Jacobi pairs with background and dual pairs

After the advent of Quantum Mechanics [25, 26], it was soon realized that Poisson geometry is the fundamental ingredient of the quantization concept [7]. This special position comes from the Poisson 2-vector which structures the set of classical observables as an algebra with respect to the Poisson bracket. Recently, the intense search for a renormalizable theory for gravitation in various space-time dimensions has given the Poisson structures new insights. These are strongly involved in various modern models of two-dimensional gravity [47, 48, 49] as well as in topological BF interacting models [8].

Nowadays, the progress of Poisson geometry knowledge has known two important milestones that have produced a lot of new mathematical results. Both directions of research have come as natural extensions of the Poisson structure. First of these was implemented by spoiling the Leibniz rule verified by the Poisson bracket through a special vector field and was done by the analysis of local Lie algebras [42] which revealed the Jacobi manifolds. This concept of Jacobi structure, via the associated Jacobi bracket, plays an important role in mathematical physics, namely in the canonical approach of non-autonomous Hamiltonian systems [81, 76], in the integrability of Hamiltonian systems on odd-dimensional manifolds [77, 78, 45] as well as in the geometric reformulation of non-equilibrium thermodynamics [5]. The starting point of the second direction grew out of string theory, by the modification of the Poisson identity through a closed 3-form (background). This relaxation has put into evidence a new class of structures, namely the twisted Poisson (or Poisson with (closed) 3-form background) [70].

A joint generalization of the above structures is the twisted Jacobi structure [62], where the obstruction to the Jacobi identity for a Jacobi bracket involves a 2-form and its de Rham differential. Concerning these new structures, it has been shown that their characteristic distributions are integrable, with a twisted contact structure on the odd dimensional leaves and a twisted locally conformal symplectic structure on the even dimensional leaves [64]. Moreover, it has been proved that the twisted Jacobi manifolds are in one-to-one correspondence with homogeneous twisted Poisson manifolds, where the background 3-form is exact [62].

At this stage, it is natural to ask if we can further relax the twisted Jacobi concept such that these extended structures i) are in a one-to-one relation with the homogeneous Poisson structures with *arbitrary* 3-form background, and ii) display integrable characteristic distributions. The aim of this chapter is to prove that the answer to the previous questions is positive, by introducing the concept of *Jacobi structure with background* (see Definition 3.1.3), where the background consists of a 3-form together with a 2-form. As shown in Theorem 3.4.7, the odd dimensional characteristic leaves are twisted contact, just like in the twisted Jacobi case. The even dimensional characteristic leaves admit a structure that is more general than the twisted locally conformal symplectic one (obtained in the twisted Jacobi case): we call it *locally conformal symplectic structure with background* (see Definition 4.4.4).

Similarly to the twisted version of a (symplectic) dual pair [37], we define and study a twisted version (in the trivial line bundle setting) of the contact dual pair [73]. Via "Poissonization", a twisted contact dual pair yields a homogeneous twisted symplectic dual pair. We show the correspondence of the characteristic leaves in the two Jacobi manifolds with background in a twisted contact dual pair that is full and with connected fibers.

It is worth noticing that the analyzed structures find their 'global' expression [55] into Jacobi bundles with background, structures that are to be approached in the next chapter.

The present chapter is organized into five sections as follows. In Section 2, we introduce and exemplify the concept of *Jacobi structure with background*. Due to the special framework–trivial line bundle, such structures consist of pairs of geometric objects (one 2-vector field and one vector field) and will be addressed as *Jacobi pairs with background*. The twisted Jacobi pairs and the Poisson structures with background are special cases of this new construct. Section 3 is dedicated to the completion of the category whose objects are the Jacobi manifolds with background (manifolds endowed with Jacobi pairs with background). Here, we adapt and investigate the notions of Jacobi map and conformal Jacobi morphism [62]. In Section 4, we prove that there is a one-to-one correspondence between Jacobi manifolds with background and homogeneous Poisson manifolds with background. Section 5 deals with the (singular) characteristic distribution associated to a Jacobi pair with background. Here, we prove that it is completely integrable and its characteristic leaves are either locally conformal symplectic manifolds with background or twisted contact manifolds. Section 6 ends the chapter with twisted dual pairs in the symplectic and contact setting. We emphasize two results, one concerning the characteristic leaf correspondence and the other about "Poissonization" of a twisted contact dual pair.

The original contribution to this chapter is contained in [16, 17, 18].

3.1 The concept

Let M be a smooth manifold. We denote by $\Omega^p(M)$ and $\mathfrak{X}^p(M)$ the spaces of smooth p-forms and smooth p-vector fields respectively, so $\Omega^0(M) = \mathcal{F}(M) = \mathfrak{X}^0(M)$, where $\mathcal{F}(M) := \mathcal{C}^\infty(M)$ represents the set of real smooth functions defined on M. We adopt the conventions from [55], concerning the wedge products, interior products, pairings between $\Omega^p(M)$ and $\mathfrak{X}^p(M)$ and also those for Schouten-Nijenhuis bracket $[\ ,\ ]$ on multi-vector fields. All of these can be found in Chapter 1 of the present work.

3.1.1 Twisted Jacobi pairs

Let M be a smooth manifold. A *Jacobi structure with trivial line bundle* [76], called a *Jacobi pair on M* in [23, 9], is a pair (Π, E) built with a bi-vector field and a vector field that satisfy

$$\tfrac{1}{2}\,[\Pi,\Pi] + E \wedge \Pi = 0, \quad [E,\Pi] = 0. \tag{3.1}$$

Equations (3.1) that define a Jacobi pair can be expressed in a more economical way, which is suitable for generalization, as we sketch in the sequel. We consider the Lie algebroid

$$(TM \times \mathbb{R}, [\![\cdot,\cdot]\!], \rho) \tag{3.2}$$

with bracket

$$[\![(X,f),(Y,g)]\!] := ([X,Y]\,, X(g) - Y(f)), \tag{3.3}$$

with anchor the projection ρ on the first factor. It can be extended to the Gerstenhaber bracket on $\Gamma\left(\wedge^\bullet (TM \times \mathbb{R})\right)$, which reads

$$[\![(P,Q),(R,S)]\!] = \left([P,R]\,, [P,S] + (-)^r\,[Q,R]\right). \tag{3.4}$$

The Gerstenhaber algebra structure on $\Gamma\left(\wedge^\bullet (TM \times \mathbb{R})\right)$ is completed by the graded commutative and associative multiplication

$$(P,Q) \wedge (R,S) = (P \wedge R, P \wedge S - (-)^r Q \wedge R), \tag{3.5}$$

where $(P,Q) \in \Gamma\left(\wedge^{p+1} (TM \times \mathbb{R})\right) \simeq \mathfrak{X}^{p+1}(M) \times \mathfrak{X}^p(M)$ and
$(R,S) \in \Gamma\left(\wedge^{r+1} (TM \times \mathbb{R})\right) \simeq \mathfrak{X}^{r+1}(M) \times \mathfrak{X}^r(M)$.

Now the two identities (3.1) can be expressed [40] in terms of the Gerstenhaber bracket $[\![\cdot,\cdot]\!]^{(0,1)}$ associated with the 1-cocycle $(0,1) \in \Gamma(T^*M \times \mathbb{R})$ as

$$\tfrac{1}{2}[\![(\Pi,E),(\Pi,E)]\!]^{(0,1)} = 0. \tag{3.6}$$

The closedness of 1-cochain $(0,1) \in \Gamma(T^*M \times \mathbb{R})$ refers to the de Rham differential associated with the Lie algebroid (3.2),

$$\mathbf{d}\,(\omega,\alpha) := (\mathrm{d}\omega, -\mathrm{d}\alpha)\,, \quad (\omega,\alpha) \in \Omega^k\,(M) \times \Omega^{k-1}\,(M)\,, \tag{3.7}$$

while the modified Gerstenhaber bracket has additional terms to (3.4):

$$\begin{aligned}[\![(P,Q),(R,S)]\!]^{(0,1)} = ([P,R] - p(-)^r P \wedge S + rQ \wedge R,&\\ [P,S] + (-)^r\,[Q,R] - (p-r)\,Q \wedge S)\,.&\end{aligned} \tag{3.8}$$

Indeed, bracket (3.8) can be written in terms of (3.4) and (3.5) as

$$\begin{aligned}[\![(P,Q),(R,S)]\!]^{(0,1)} = [\![(P,Q),(R,S)]\!] + p(-)^r (P,Q) \wedge j_{(0,1)}(R,S)&\\ - r j_{(0,1)}(P,Q) \wedge (R,S)&\end{aligned} \tag{3.9}$$

with

$$j_{(0,1)}(P,Q) = -(Q,0).$$

It is noteworthy that the homological degree 1 derivation acting on the module $\Gamma\left(\wedge^\bullet(T^*M\times\mathbb{R})\right)$ (over $\Omega^\bullet(M)$) associated with the modified Gerstenhaber bracket (3.8) reads

$$\mathbf{d}^{(0,1)}(\omega,\alpha) = (\mathrm{d}\omega, -\mathrm{d}\alpha+\omega), \quad (\omega,\alpha)\in\Omega^k(M)\times\Omega^{k-1}(M). \tag{3.10}$$

The notion of twisted Jacobi manifold (with trivial line bundle) was introduced in [62] as follows.

Definition 3.1.1. [62] *Let M be a smooth manifold. The structure $((\Pi,E),\omega)$, consisting in*

$$\Pi\in\mathfrak{X}^2(M), \quad E\in\mathfrak{X}^1(M), \quad \omega\in\Omega^2(M)$$

and enjoying the properties

$$\tfrac{1}{2}[\Pi,\Pi]+E\wedge\Pi = \Pi^\sharp\mathrm{d}\omega+\Pi^\sharp\omega\wedge E, \quad [E,\Pi] = -\left(\Pi^\sharp i_E\mathrm{d}\omega+\Pi^\sharp i_E\omega\wedge E\right), \tag{3.11}$$

is called a Jacobi pair (Π,E) twisted by the 2-form ω, or, simply, a twisted Jacobi pair.

In the previous formulae we use the notation

$$\Pi^\sharp:\Omega^1(M)\to\mathfrak{X}^1(M), \quad \Pi^\sharp\alpha := -j_\alpha\Pi \tag{3.12}$$

with j_α the left interior product by 1-form α. We also use the same symbol for its extension (by $\mathbb{R}$- and $\mathcal{F}(M)$-linearity) to $\Pi^\sharp:\Omega^p(M)\to\mathfrak{X}^p(M)$.

Remark 3.1.2. [62] It is worth noticing that relations (3.11) are equivalent to twisted Maurer-Cartan equation (3.6)

$$\tfrac{1}{2}[\![(\Pi,E),(\Pi,E)]\!]^{(0,1)} = (\Pi,E)^\sharp\left(\mathbf{d}^{(0,1)}(\omega,0)\right),$$

where the $\mathcal{F}(M)$-module morphism

$$(\Pi,E)^\sharp:\Gamma(\wedge^k(T^*M\times\mathbb{R}))\to\Gamma(\wedge^k(TM\times\mathbb{R})) \tag{3.13}$$

is the linear extension of

$$\Gamma(T^*M\times\mathbb{R})\ni(\beta,f)\overset{(\Pi,E)^\sharp}{\mapsto}(\Pi^\sharp(\beta)+fE,-i_E\beta)\in\Gamma(TM\times\mathbb{R}).$$

By direct computation, it results that the concrete expression of vector bundle map (3.13) reads

$$(\omega,\theta)\overset{(\Pi,E)^\sharp}{\mapsto}(\Pi^\sharp(\omega)-(-)^k\Pi^\sharp(\theta)\wedge E,\Pi^\sharp(i_E\omega)-(-)^k\Pi^\sharp(i_E\theta)\wedge E,),$$

for any pair

$$(\omega,\theta)\in\Omega^k(M)\times\Omega^{k-1}(M)\simeq\Gamma(\wedge^k(T^*M\times\mathbb{R})).$$

3.1.2 Jacobi pairs with background

By definition, a *twisted Poisson structure*, also called a *Poisson structure with (closed)* 3*-form background* [70] consists of a 2-vector field Π and a closed 3-form ϕ,

$$\mathrm{d}\phi = 0, \tag{3.14}$$

connected via

$$\tfrac{1}{2}\,[\Pi,\Pi] = \Pi^\sharp\phi. \tag{3.15}$$

It is said that the Poisson structure Π is twisted by the 3-form ϕ. The closedness condition (3.14) allows the construction of a Courant algebroid structure on $TM \oplus T^*M$, $((\bullet,\bullet),[\bullet,\bullet]_\phi,\rho)$, starting from $((\bullet,\bullet),[\bullet,\bullet],\rho)$, which structures $TM \oplus T^*M$ as a Courant algebroid. Previously, we invoked the data

$$\begin{aligned}
(X\oplus\alpha, Y\oplus\beta) &:= \frac{1}{2}(i_X\beta + i_Y\alpha)\\
[X\oplus\alpha, Y\oplus\beta] &:= [X,Y]\oplus(\mathcal{L}_X\beta - i_Y\mathrm{d}\alpha)\\
[X\oplus\alpha, Y\oplus\beta]_\phi &:= [X,Y]\oplus(\mathcal{L}_X\beta - i_Y\mathrm{d}\alpha - i_Xi_Y\phi)\\
\rho(X\oplus\alpha) &:= \alpha.
\end{aligned}$$

In this Courant algebroid with twisted Dorfman bracket, $[\bullet,\bullet]_\phi$, the graph of the twisted Poisson structure is a Dirac structure, which particularly implies its integrability [70].

In the sequel, we take into account Poisson structures with *arbitrary* 3-form background that are shortly addressed as *Poisson structures with background.* Such structure has been recently proposed in the context of sigma-models [13]. Other instances that display Poisson structures with background come from quasi Poisson concept [1] (see Section 3.1.5 below). In the light of this relaxation, we propose an extension of the Definition 3.1.1 to a Jacobi structure with background.

Definition 3.1.3. *The structure* $((\Pi,E),(\phi,\omega))$ *consisting of*

$$\Pi\in\mathfrak{X}^2(M),\quad E\in\mathfrak{X}^1(M),\quad \phi\in\Omega^3(M),\quad \omega\in\Omega^2(M),$$

defines a Jacobi pair (Π,E) *with background* (ϕ,ω) *on* M *if it satisfies*

$$\tfrac{1}{2}\,[\Pi,\Pi] + E\wedge\Pi = \Pi^\sharp\phi + \Pi^\sharp\omega\wedge E,\quad [E,\Pi] = -\left(\Pi^\sharp i_E\phi + \Pi^\sharp i_E\omega\wedge E\right). \tag{3.16}$$

It is immediate that, if in (3.16) we consider $E=0$ and $\omega=0$, we get nothing but a Poisson structure with background (3.15). Moreover, if the background is of the form

$$(\phi,\omega) = \mathbf{d}^{(0,1)}(\omega,0) := (\mathrm{d}\omega,\omega),$$

we obtain the well-known twisted Jacobi structures (see Definition 3.1.1). It is noteworthy that, by means of definition (3.10), the previous condition is equivalent to $\mathbf{d}^{(0,1)}$-closedness of the pair (ϕ,ω).

Remark 3.1.4. In terms of the Lie algebroid (3.2) with 1-cocycle $(0,1) \in \Gamma(T^*M \times \mathbb{R})$ and the modified Gerstenhaber bracket (3.8), the equations (3.16) are equivalent to

$$\tfrac{1}{2}[\![(\Pi, E), (\Pi, E)]\!]^{(0,1)} = (\Pi, E)^\sharp(\phi, \omega).$$

In the same language, due to the fact that the homological degree 1 derivation $\mathbf{d}^{(0,1)}$ in (3.10) is acyclic, it results that twisted Jacobi structures are nothing but Jacobi structures with closed background

$$\mathbf{d}^{(0,1)}(\phi, \omega) = 0. \tag{3.17}$$

According to [62], the Dorfman bracket of the standard Courant-Jacobi algebroid structure on the omni Lie algebroid $(TM \times \mathbb{R}) \oplus (T^*M \times \mathbb{R})$ [80] can be twisted by (ϕ, ω) if and only if (3.17) holds.

In view of evidencing the existence of such a structure, we give some illustrative examples.

Example 3.1.5. Let's consider the four-dimensional smooth manifold $\mathbb{R}^4$ with the global coordinates $x = (x^1, x^2, x^3, x^4)$ and the real smooth functions $a, b \in \mathcal{F}(\mathbb{R}^4)$ among which a is nowhere vanishing and b depends only on the first two coordinates $b(x^1, x^2)$. The geometric objects

$$\begin{aligned} &\Pi = a^{-1}(\partial_1 \wedge \partial_4 + \partial_2 \wedge \partial_3), \quad E = -a^{-1}((\partial_1 b)\,\partial_4 + (\partial_2 b)\,\partial_3), \\ &\phi = \mathrm{d}\left(a\ \mathrm{d}x^2 \wedge \mathrm{d}x^3 + a\ \mathrm{d}x^1 \wedge \mathrm{d}x^4\right) - a\mathrm{d}\left(b\ \mathrm{d}x^2 \wedge \mathrm{d}x^3 + b\ \mathrm{d}x^1 \wedge \mathrm{d}x^4\right), \quad \omega = 0 \end{aligned}$$

satisfy the equations (3.16) so it organizes $\mathbb{R}^4$ as a Jacobi manifold with background, with non-closed 3-form ϕ and trivial twisting 2-form ω.

Example 3.1.6. Again, we consider the four-dimensional smooth manifold $\mathbb{R}^4$ and we take functions $a, b \in \mathcal{F}(\mathbb{R}^4)$ with a nowhere vanishing. We globally define the geometric objects

$$\Omega = a\left(\mathrm{d}x^1 \wedge \mathrm{d}x^2 + \mathrm{d}x^3 \wedge \mathrm{d}x^4\right), \quad \phi = \mathrm{d}\omega + (\mathrm{d}a + a\ \mathrm{d}b) \wedge \mathrm{d}x^3 \wedge \mathrm{d}x^4, \quad \omega = a\ \mathrm{d}x^1 \wedge \mathrm{d}x^2. \tag{3.18}$$

The 2-form Ω is non-degenerate with inverse the bi-vector field Π given by

$$\langle \rho \wedge \lambda, \Pi \rangle := \left\langle \Omega, \Omega^\sharp \rho \wedge \Omega^\sharp \lambda \right\rangle, \quad \rho, \lambda \in \Omega^1(M), \tag{3.19}$$

thus $\Pi = -a^{-1}(\partial_1 \wedge \partial_2 + \partial_3 \wedge \partial_4)$. This allows the construction of the vector field $E = \Pi^\sharp \mathrm{d}b$. With these tools at hand, the pair $((\Pi, E), (\phi, \omega))$ organizes $\mathbb{R}^4$ as a Jacobi manifold with background. Here, the twisting 2-form ω is non-trivial.

Example 3.1.7. *Let's consider the four-dimensional smooth manifold $\mathbb{R}^4$ with the global coordinates $x = (x^1, x^2, x^3, x^4)$ and the real smooth functions $f, e \in \mathcal{F}(\mathbb{R}^4)$ among which f is nowhere vanishing, $f^2 > 0$, and e depends only on the first two coordinates, $e = e(x^1, x^2)$. We introduce the geometric objects*

$$\Pi = \frac{1}{f}(\partial_1 \wedge \partial_4 + \partial_2 \wedge \partial_3), \qquad E = -\frac{1}{f}((\partial_1 e)\,\partial_4 + (\partial_2 e)\,\partial_3), \tag{3.20}$$

$$\phi = (\mathrm{d}f - f\,de) \wedge (x^2 \wedge \mathrm{d}x^3 + \mathrm{d}x^1 \wedge \mathrm{d}x^4), \qquad \omega = 0. \tag{3.21}$$

Direct computations show that the geometric objects (3.20) and (3.21) verify the relations (3.16) *which means that* $((\Pi, E), (\phi, \omega))$ *is a Jacobi pair with background.*

Example 3.1.8. *Let's consider again the four-dimensional smooth manifold $\mathbb{R}^4$ with the global coordinates $x = (x^1, x^2, x^3, x^4)$ and the smooth functions f, g, h, and e on $\mathbb{R}^4$ among which f, and e verify the same restrictions as in the previous example. Construct the bi-vector and vector fields as in* (3.20) *and*

$$\begin{aligned}\phi &= \left[(\partial_3 h \partial_2 e + \partial_4 f)\,\mathrm{d}x^2 + (\partial_3 h \partial_1 e - \partial_3 f)\,\mathrm{d}x^1\right] \wedge \mathrm{d}x^3 \wedge \mathrm{d}x^4 \\ &\quad + \left[(f - \partial_2 f \partial_2 e - \partial_1 h \partial_2 e + \partial_2 h \partial_1 e)\,\mathrm{d}x^4 + \left(\partial_1 f + \partial_3 \tilde{f} \partial_2 e\right)\mathrm{d}x^3\right] \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2, \qquad (3.22)\end{aligned}$$

$$\omega = f(x)\,\mathrm{d}x^2 \wedge \mathrm{d}x^3 + \left(\partial_3 \tilde{f}\right)\mathrm{d}x^3 \wedge \mathrm{d}x^1 + g(x)\,\mathrm{d}x^1 \wedge \mathrm{d}x^2 + \mathrm{d}h \wedge \mathrm{d}x^4. \qquad (3.23)$$

In formulas (3.22) and (3.23) $\tilde{f}$ stands for an arbitrary smooth function that verifies $\partial_4 \tilde{f} = f$. By direct computation it can be checked that the pair $((\Pi, E), (\phi, \omega))$ given in (3.20), (3.22), and (3.23) satisfies the compatibility conditions (3.16)*, i.e. it is a Jacobi pair with background. Simple computations show that the* 2*-form ω is non-trivial while the* 3*-form is closed but*

$$\phi \neq \mathrm{d}\omega.$$

3.1.3 Locally conformal symplectic structures with background

Definition 3.1.9. *Let M be an even-dimensional smooth manifold endowed with a non-degenerate* 2*-form Ω. Given α a closed* 1*-form (the Lee form)*

$$\mathrm{d}\alpha = 0, \qquad (3.24)$$

and ω a 2*-form, we define the 3-form*

$$\phi := \mathrm{d}\Omega + \alpha \wedge (\Omega - \omega). \qquad (3.25)$$

The structure $((\Omega, \alpha), (\phi, \omega))$ on M is said to be a locally conformal symplectic structure (Ω, α) with background (ϕ, ω).

The previous examples 3.1.7 and 3.1.6 exhibit such triplets, so they are locally conformal symplectic structures with background.

Remark 3.1.10. If the Lee 1-form α vanishes, we have

$$\phi = \mathrm{d}\Omega,$$

so we obtain the notion of *twisted symplectic structure* (also known under the name *almost symplectic structure*) [36]. It induces a twisted Poisson structure (Π, ϕ) with Π the inverse of Ω [68]. Another special case is

$$\phi = \mathrm{d}\omega,$$

when we recover the notion of *twisted locally conformal symplectic structure* [64], since

$$d(\Omega - \omega) + \alpha \wedge (\Omega - \omega) = 0.$$

Proposition 3.1.11. *A locally conformal symplectic structure with background* $((\Omega,\alpha),(\phi,\omega))$ *on the smooth manifold* M *naturally organizes* M *as a Jacobi manifold with background with* Π *the inverse of* Ω (3.19) *and* $E:=\Omega^{\sharp}\alpha$.

Proof. Using the Koszul relation (that connects right inner multiplications by p-vectors and Schouten bracket) [55]

$$i_{[P,Q]}=-\left[\left[i_Q,\mathrm{d}\right],i_P\right],\quad P\in\mathfrak{X}^p\left(M\right),Q\in\mathfrak{X}^q\left(M\right)\tag{3.26}$$

and definition (3.19), by direct computation we get

$$\begin{aligned}i_{\frac{1}{2}[\Pi,\Pi]}\left(\rho\wedge\lambda\wedge\mu\right)&=-i_\Pi\mathrm{d}i_\Pi\left(\rho\wedge\lambda\wedge\mu\right)=\sum_{\text{cyclic}}i_\Pi\left[\rho\wedge\mathrm{d}i_\Pi\left(\lambda\wedge\mu\right)\right]\\&=\sum_{\text{cyclic}}\Omega^\sharp\rho\left(\langle\Omega,\Omega^\sharp\lambda\wedge\Omega^\sharp\mu\rangle\right)\end{aligned}\tag{3.27}$$

for arbitrary closed 1-forms ρ, λ, and μ. Using (3.19), it further leads to

$$i_{E\wedge\Pi}\left(\rho\wedge\lambda\wedge\mu\right)=i_\Pi i_E\left(\rho\wedge\lambda\wedge\mu\right)=\sum_{\text{cyclic}}i_E\rho i_\Pi\left(\lambda\wedge\mu\right)=\sum_{\text{cyclic}}i_E\rho\langle\Omega,\Omega^\sharp\lambda\wedge\Omega^\sharp\mu\rangle.\tag{3.28}$$

Combining the relations (3.25) and (3.19), by direct computations we establish

$$\begin{aligned}i_{\Pi^\sharp\phi}\left(\rho\wedge\lambda\wedge\mu\right)&=-\langle\phi,\Pi^\sharp\rho\wedge\Pi^\sharp\lambda\wedge\Pi^\sharp\mu\rangle=-\langle\mathrm{d}\Omega,\Omega^\sharp\rho\wedge\Omega^\sharp\lambda\wedge\Omega^\sharp\mu\rangle\\&\quad-\langle\alpha\wedge\Omega,\Omega^\sharp\rho\wedge\Omega^\sharp\lambda\wedge\Omega^\sharp\mu\rangle+\langle\alpha\wedge\omega,\Omega^\sharp\rho\wedge\Omega^\sharp\lambda\wedge\Omega^\sharp\mu\rangle\\&=\sum_{\text{cyclic}}\Omega^\sharp\rho\left(\langle\Omega,\Omega^\sharp\lambda\wedge\Omega^\sharp\mu\rangle\right)+i_E\rho\langle\Omega,\Omega^\sharp\lambda\wedge\Omega^\sharp\mu\rangle-i_E\rho\langle\omega,\Omega^\sharp\lambda\wedge\Omega^\sharp\mu\rangle\end{aligned}\tag{3.29}$$

and

$$i_{\Pi^\sharp\omega\wedge E}\left(\rho\wedge\lambda\wedge\mu\right)=i_Ei_{\Pi^\sharp\omega}\left(\rho\wedge\lambda\wedge\mu\right)=\sum_{\text{cyclic}}i_E\rho\left\langle\omega,\Omega^\sharp\lambda\wedge\Omega^\sharp\mu\right\rangle.\tag{3.30}$$

The results (3.27)–(3.30) allow us to conclude that the first relation in (3.16) holds.

Using again formula (3.26), we get for all $\rho,\lambda\in\Lambda^1(M)$:

$$\begin{aligned}i_{[E,\Pi]}(\rho\wedge\lambda)&=-i_\Pi\mathrm{d}i_E\left(\rho\wedge\lambda\right)+i_E\mathrm{d}i_\Pi\left(\rho\wedge\lambda\right)\\&=-\Omega^\sharp\rho\left(\langle\lambda,E\rangle\right)+\Omega^\sharp\lambda\left(\langle\rho,E\rangle\right)-E\left(\langle\Omega,\Omega^\sharp\rho\wedge\Omega^\sharp\lambda\rangle\right).\end{aligned}\tag{3.31}$$

In addition, direct computations based on the definitions (3.25) and (3.19) reveal

$$\begin{aligned}i_{\Pi^\sharp i_E\phi}\left(\rho\wedge\lambda\right)&=\left\langle\mathrm{d}\Omega,E\wedge\Omega^\sharp\rho\wedge\Omega^\sharp\lambda\right\rangle-i_E\rho\left\langle\Omega,\Omega^\sharp\lambda\wedge E\right\rangle+i_E\lambda\left\langle\Omega,\Omega^\sharp\rho\wedge E\right\rangle\\&\quad+i_E\rho\langle\omega,\Omega^\sharp\lambda\wedge E\rangle-i_E\lambda\langle\omega,\Omega^\sharp\rho\wedge E\rangle\\&=\Omega^\sharp\rho\left(\langle\lambda,E\rangle\right)-\Omega^\sharp\lambda\left(\langle\rho,E\rangle\right)+E\left(\langle\Omega,\Omega^\sharp\rho\wedge\Omega^\sharp\lambda\rangle\right)\\&\quad+i_E\rho\langle\omega,\Omega^\sharp\lambda\wedge E\rangle-i_E\lambda\langle\omega,\Omega^\sharp\rho\wedge E\rangle\end{aligned}\tag{3.32}$$

and

$$i_{\Pi^\sharp i_E\omega\wedge E}\left(\rho\wedge\lambda\right)=i_Ei_{\Pi^\sharp i_E\omega}\left(\rho\wedge\lambda\right)=-i_E\rho\left\langle\omega,\Omega^\sharp\lambda\wedge E\right\rangle+i_E\lambda\left\langle\omega,\Omega^\sharp\rho\wedge E\right\rangle.\tag{3.33}$$

Finally, the equalities (3.31)–(3.33) prove that the second identity in (3.16) also takes place.

□

3.1.4 Twisted contact structures

Let M be a $(2m+1)$-dimensional smooth manifold. The pair of forms (θ, ω) with $\omega \in \Omega^2(M)$ and $\theta \in \Omega^1(M)$, such that

$$\mu := \theta \wedge (\mathrm{d}\theta + \omega)^m \neq 0 \tag{3.34}$$

is a volume form on M, is said to be a *twisted (co-orientable) contact structure* [62].

The Hamiltonian vector field to $f \in \mathcal{F}(M)$ is defined as the unique solution X_f to

$$i_{X_f}\theta = f, \quad i_{X_f}(\mathrm{d}\theta + \omega) = i_E\left(\mathrm{d}f \wedge \theta\right), \tag{3.35}$$

Proposition 3.1.12. *Let (θ, ω) be a twisted contact structure on the smooth manifold M. Then it naturally organizes M as a twisted Jacobi manifold with respect to $((\Pi, E), \omega)$, where E is the unique vector field (Reeb vector field) that satisfies*

$$i_E\theta = 1, \quad i_E(\mathrm{d}\theta + \omega) = 0, \tag{3.36}$$

and

$$i_\Pi\left(\mathrm{d}f \wedge \mathrm{d}g\right) := \langle \mathrm{d}\theta + \omega, X_f \wedge X_g\rangle, \quad f, g \in \mathcal{F}(M). \tag{3.37}$$

Proof. Employing the equations (3.36)–(3.35), it can be shown that the 2-vector field Π given by (3.37) verifies

$$i_{\Pi^\sharp\alpha}\theta = 0, \quad i_{\Pi^\sharp\alpha}\left(\mathrm{d}\theta + \omega\right) = i_E\left(\alpha \wedge \theta\right) = -\alpha + (i_E\alpha)\theta, \quad \alpha \in \Omega^1(M). \tag{3.38}$$

By means of these results, direct computations show that the structure $((\Pi, E), \omega))$ is twisted Jacobi. It is worth noticing that this output has been exhibited in [64] where the bivector Π is defined via (3.38). □

It is immediate that the 2-form $\Omega = \mathrm{d}\theta + \omega$ restricted to the hyperplane distribution $\mathcal{H} := \ker\theta$ is non-degenerate. Indeed, taking into account the second indentity in (3.38) and using a dimension count we deduce an expression for the orthogonality on $\mathcal{H}$ with respect to Ω:

$$V^\perp = \Pi^\sharp(V^\circ), \quad V \subseteq \mathcal{H}, \tag{3.39}$$

where $V^\circ \subseteq T^*M$ denotes the annihilator of V.

3.1.5 Quasi Poisson/Jacobi structures

A *quasi Poisson structure* [1] on the manifold M comes with a Lie algebra action, i.e. a Lie algebra homomorphism $a : \mathfrak{g} \to \mathfrak{X}(M)$, and an invariant inner product $\cdot$ on $\mathfrak{g}$, such that

$$\tfrac{1}{2}\left[\Pi, \Pi\right] = a(\chi), \quad \mathcal{L}_{a(\xi)}\Pi = [a(\xi), \Pi] = 0, \quad \xi \in \mathfrak{g}, \tag{3.40}$$

where $\chi \in \Lambda^3\mathfrak{g}$ denotes the Cartan 3-tensor. If the $\mathfrak{g}$-action is tangent to the characteristic distribution $\operatorname{Im}\Pi^\sharp$, then $a(\chi) = \Pi^\sharp\phi$ with $\phi \in \Omega^3(M)$ and the quasi Poisson structure induces a Poisson structure with background that is a not necessarily closed 3-form ϕ.

A *quasi Jacobi structure* [63] involves an additional ingredient: a Lie algebra 1-cocyle $\lambda \in \mathfrak{g}^*$, which means that λ vanishes on $[\mathfrak{g}, \mathfrak{g}]$. The Lie algebra homomorphism $a^\lambda : \mathfrak{g} \to$

$\Gamma(TM \times \mathbb{R})$ with respect to the Lie algebroid bracket (3.3), given by $a^\lambda(\xi) = (a(\xi), \lambda(\xi))$, is called a $\mathfrak{g}$-action on M with 1-cocycle λ [63]. Similarly to (3.40), by definition a quasi Jacobi structure satisfies:

$$\tfrac{1}{2}[\![(\Pi, E), (\Pi, E)]\!]^{(0,1)} = a^\lambda(\chi), \quad \mathcal{L}^{(0,1)}_{a^\lambda(\xi)}(\Pi, E) = 0, \quad \xi \in \mathfrak{g}. \tag{3.41}$$

If the Lie algebra action with 1-cocycle is tangent to the distribution $\mathrm{Im}(\Pi, E)^\sharp \subseteq TM \times \mathbb{R}$, then

$$a^\lambda(\chi) = (\Pi, E)^\sharp(\phi, \omega) \in \Gamma(\Lambda^3(TM \times \mathbb{R})) \tag{3.42}$$

with $\phi \in \Omega^3(M)$ and $\omega \in \Omega^2(M)$. In this case the quasi Jacobi structure induces a more general structure than the twisted-Jacobi one, since ϕ is not necessarily equal to $\mathrm{d}\omega$, thus a Jacobi structure with background.

The Cartan 3-tensor defined by the invariant inner product $\cdot$ on $\mathfrak{g}$ is

$$\chi = \tfrac{1}{12} f_{abc} e_a \wedge e_b \wedge e_c \in \Lambda^3 \mathfrak{g},$$

where $\{e_a\}$ is an orthonormal basis and $f_{abc} = e_a \cdot [e_b, e_c]$. Thus the 3-vector component of $a^\lambda(\chi)$ is

$$\tfrac{1}{12} f_{abc} a(e_a) \wedge a(e_b) \wedge a(e_c) = a(\chi).$$

On the other hand the 2-vector component of $a^\lambda(\chi)$,

$$\tfrac{1}{12} f_{abc} \left(\lambda(e_a) a(e_b) \wedge a(e_c) + \lambda(e_b) a(e_c) \wedge a(e_a) + \lambda(e_c) a(e_a) \wedge a(e_b)\right),$$

vanishes because, by the 1-cocycle condition for $\lambda \in \mathfrak{g}^*$, we have for all indices b, c:

$$f_{abc} \lambda(e_a) = \lambda((e_a \cdot [e_b, e_c]) e_a) = \lambda([e_b, e_c]) = 0.$$

Thus, in the definition (3.41) of a quasi-Jacobi manifold, we actually have:

$$a^\lambda(\chi) = (a(\chi), 0) \in \Gamma(\Lambda^3(TM \times \mathbb{R})) = \mathfrak{X}^3(M) \times \mathfrak{X}^2(M).$$

Because the expression of (3.13) for $k = 3$ is

$$(\Pi, E)^\sharp(\phi, \omega) = (\Pi^\sharp \phi + \Pi^\sharp \omega \wedge E, -\Pi^\sharp i_E \phi - \Pi^\sharp i_E \omega \wedge E),$$

from the identity (3.42) we deduce an additional condition on the background forms (ϕ, ω) on a quasi Jacobi manifold:

$$\Pi^\sharp i_E \phi + \Pi^\sharp i_E \omega \wedge E = 0.$$

3.2 Brackets for Jacobi structures with background

3.2.1 Brackets and Hamiltonian vector fields

Starting with a Jacobi manifold with background $(M, ((\Pi, E), (\phi, \omega)))$ and using the same type of Jacobi bracket as for twisted Jacobi manifolds [62], we endow the set of smooth functions $\mathcal{F}(M)$ with the bracket

$$\{f, g\} := i_\Pi (\mathrm{d}f \wedge \mathrm{d}g) + i_E (f \mathrm{d}g - g \mathrm{d}f), \quad f, g \in \mathcal{F}(M). \tag{3.43}$$

Definition (3.43) displays a skew-symmetric, bi-differential operator of the $\mathcal{F}(M)$-module $\mathcal{F}(M)$

$$\{f, gh\} = (X_f g)\, h + g\, \{f, h\}\,, \quad f, g, h \in \mathcal{F}(M),$$

where

$$X_f := \Pi^\sharp \mathrm{d}f + fE \tag{3.44}$$

is *the Hamiltonian vector field* corresponding to the smooth function f. In the special case of a twisted contact manifold, this Hamiltonian vector field coincides with the one given in (3.35).

The bracket (3.43) can be also expressed using Hamiltonian vector fields as

$$\{f, g\} = i_{X_f}\mathrm{d}g - g i_E \mathrm{d}f. \tag{3.45}$$

In general, this twisted Jacobi bracket doesn't satisfy the Jacobi identity.

Proposition 3.2.1. *The bracket* (3.43) *satisfies*

$$\mathrm{Jac}\,\{f, g, h\} = -i_{X_f\wedge X_g\wedge X_h}\phi + (\mathcal{L}_E f) i_{X_g\wedge X_h}\omega + (\mathcal{L}_E g) i_{X_h\wedge X_f}\omega + (\mathcal{L}_E h) i_{X_f\wedge X_g}\omega, \tag{3.46}$$

with the Jacobiator defined by

$$\mathrm{Jac}\,\{f, g, h\} := \{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\}\,.$$

Proof. By means of definition (3.43), direct computations display

$$\begin{aligned}\{f, \{g, h\}\} = &-i_\Pi \mathrm{d}\left(i_\Pi\left(\mathrm{d}g\wedge\mathrm{d}h\right)\mathrm{d}f\right) + i_\Pi\left(\mathrm{d}f\wedge\mathcal{L}_E\left(g\mathrm{d}h - h\mathrm{d}g\right)\right)\\ &- 2 i_\Pi\left(\mathrm{d}f\wedge i_E\left(\mathrm{d}g\wedge\mathrm{d}h\right)\right) + f\mathcal{L}_E\left(i_\Pi\left(\mathrm{d}g\wedge\mathrm{d}h\right) + i_E\left(g\mathrm{d}h - h\mathrm{d}g\right)\right)\\ &- \mathcal{L}_E f\left(i_\Pi\left(\mathrm{d}g\wedge\mathrm{d}h\right) + g\mathcal{L}_E h - h\mathcal{L}_E g\right).\end{aligned} \tag{3.47}$$

By considering the cyclic permutations of result (3.47), we get

$$\begin{aligned}\mathrm{Jac}\,\{f, g, h\} = &-i_\Pi \mathrm{d} i_\Pi\left(\mathrm{d}f\wedge\mathrm{d}g\wedge\mathrm{d}h\right) + i_\Pi i_E\left(\mathrm{d}f\wedge\mathrm{d}g\wedge\mathrm{d}h\right)\\ &- \sum_{\text{cyclic}}\left(i_\Pi\mathcal{L}_E\left(f\mathrm{d}g\wedge\mathrm{d}h\right) - i_E \mathrm{d} i_\Pi\left(f\mathrm{d}g\wedge\mathrm{d}h\right)\right).\end{aligned} \tag{3.48}$$

At this stage, Koszul relation (3.26) expresses the previous result into

$$\begin{aligned}\mathrm{Jac}\,\{f, g, h\} = &\, i_{\frac{1}{2}[\Pi,\Pi]}\left(\mathrm{d}f\wedge\mathrm{d}g\wedge\mathrm{d}h\right) + i_{E\wedge\Pi}\left(\mathrm{d}f\wedge\mathrm{d}g\wedge\mathrm{d}h\right)\\ &+ i_{[E,\Pi]}\left(f\mathrm{d}g\wedge\mathrm{d}h + g\mathrm{d}h\wedge\mathrm{d}f + h\mathrm{d}f\wedge\mathrm{d}g\right),\end{aligned} \tag{3.49}$$

that, by means of identities (3.16), eventually proves (3.46). □

3.2.2 Conformal factors

Adapting the construction of conformal related Jacobi structures [33], [24] to the current context, it can be shown that for a given manifold M equipped with a Jacobi structure with background $((\Pi, E), (\phi, \omega))$ there exists a family of Jacobi structures with background $((\Pi^a, E^a), (\phi^a, \omega^a))$ parameterized by smooth nowhere vanishing functions a on M, such that the corresponding brackets are *conformally related*

$$\{f, g\}^a = a^{-1} \{af, ag\} . \tag{3.50}$$

Proposition 3.2.2. *Given a Jacobi structure with background* $((\Pi, E), (\phi, \omega))$ *on* M *and a nowhere vanishing function* $a \in \mathcal{F}(M)$*,*

$$\Pi^a = a\Pi, \quad E^a = aE + \Pi^\sharp \mathrm{d}a, \quad \phi^a = a^{-1}\phi + \mathrm{d}\left(a^{-1}\right) \wedge \omega, \quad \omega^a = a^{-1}\omega. \tag{3.51}$$

define another Jacobi structure with background on M*.*

The two structures are called *conformally related Jacobi structures with background.* Notice that $E^a = X_a$, the Hamiltonian vector field for $a \in \mathcal{F}(M)$.

If the original structure $((\Pi, E), (\phi, \omega))$ is twisted Jacobi, i.e. $\phi = \mathrm{d}\omega$, then the conformally related one $((\Pi^a, E^a), (\phi^a, \omega^a))$ is also a twisted Jacobi, i.e. $\phi^a = \mathrm{d}\omega^a$. It is worth noticing that the conformally related twisted Jacobi structures have been explicitly given in [62] and [65].

Proof. The Schouten-Nijenhuis bracket

$$[\Pi^a, \Pi^a] = -2a\Pi^\sharp \mathrm{d}a \wedge \Pi + a^2 [\Pi, \Pi] ,$$

further gives

$$\tfrac{1}{2} [\Pi^a, \Pi^a] + E^a \wedge \Pi^a = a^2 \left(\Pi^\sharp\phi + \Pi^\sharp\omega \wedge E\right) . \tag{3.52}$$

On the other hand

$$\begin{aligned}(\Pi^a)^\sharp\phi^a + (\Pi^a)^\sharp\omega^a \wedge E^a &= a^3\Pi^\sharp(a^{-1}\phi + \mathrm{d}\left(a^{-1}\right) \wedge \omega) + a^2\Pi^\sharp(a^{-1}\omega) \wedge (aE + \Pi^\sharp \mathrm{d}a) \\ &= a^2 \left(\Pi^\sharp\phi + \Pi^\sharp\omega \wedge E\right) ,\end{aligned}$$

so the first identity in (3.16) is satisfied. The second one is obtained similarly. □

Proposition 3.2.3. *The corresponding brackets of two conformally related Jacobi structures with background* $((\Pi, E), (\phi, \omega))$ *and* $((\Pi^a, E^a), (\phi^a, \omega^a))$ *satisfy* (3.50)*.*

Proof. Combining definition (3.43) with the relations (3.51), we get

$$\begin{aligned}\{f, g\}^a &= i_{a\Pi} (\mathrm{d}f \wedge \mathrm{d}g) + i_{aE+\Pi^\sharp \mathrm{d}a} (f\mathrm{d}g - g\mathrm{d}f) \\ &= ai_\Pi (\mathrm{d}f \wedge \mathrm{d}g) + i_\Pi (\mathrm{d}a \wedge (f\mathrm{d}g - g\mathrm{d}f)) + ai_E (f\mathrm{d}g - g\mathrm{d}f) \\ &= a^{-1}i_\Pi (\mathrm{d}(af) \wedge \mathrm{d}(ag)) + a^{-1}i_E ((af)\mathrm{d}(ag) - (ag)\mathrm{d}(af)) = a^{-1}\{af, ag\},\end{aligned}$$

the requested relation between the conformally related brackets. □

3.2.3 Jacobi maps

At this stage, we complete with morphisms the category of Jacobi manifolds with background.

Definition 3.2.4. *Let* $(M_i, ((\Pi_i, E_i), (\phi_i, \omega_i)))$*,* $i = 1, 2$*, be two Jacobi manifolds with background. A smooth map* $F : M_1 \to M_2$ *is said to be a Jacobi map if and only if*

$$F^* \{f, g\}_2 = \{F^* f, F^* g\}_1, \quad f, g \in \mathcal{F}(M_2). \tag{3.53}$$

Proposition 3.2.5. *In the stated context, the smooth map* $F : M_1 \to M_2$ *is a Jacobi map between Jacobi manifolds with background if and only if the pair of vector fields* (E_1, E_2) *and the pair of bi-vector fields* (Π_1, Π_2) *are* F*-related. i.e.*

$$\Pi_2 \circ F = TF \circ \Pi_1, \quad E_2 \circ F = TF \circ E_1, \tag{3.54}$$

where we denote by TF *both the tangent map and its extension (by linearity) to p-vectors.*

Proof. Let f and g be two arbitrary smooth functions on M_2. We successively establish

$$\begin{aligned} \{F^* f, F^* g\}_1 &= i_{\Pi_1} F^* (\mathrm{d}f \wedge \mathrm{d}g) + i_{E_1} F^* (f \mathrm{d}g - g \mathrm{d}f) \\ &= F^* \left(i_{TF(\Pi_1)} (\mathrm{d}f \wedge \mathrm{d}g) + i_{TF(E_1)} (f \mathrm{d}g - g \mathrm{d}f) \right). \end{aligned}$$

Furthermore, definition (3.43) exhibits

$$F^* \{f, g\}_2 = F^* \left(i_{\Pi_2} (\mathrm{d}f \wedge \mathrm{d}g) + i_{E_2} (f \mathrm{d}g - g \mathrm{d}f) \right).$$

In the light of the last two results, it is clear that (3.53) holds if and only if (3.54) take place. □

Remark 3.2.6. It can be shown that the previous smooth map F is a Jacobi map if and only if the Hamiltonian vector fields $(X_{F^* f}, X_f)$ are F-related vector fields for all $f \in \mathcal{F}(M_2)$, i.e.

$$TF(X_{F^* f}) = X_f \circ F, \quad f \in \mathcal{F}(M_2). \tag{3.55}$$

Indeed, the definition of Hamiltonian vector fields (3.44) displays

$$X_{F^* f} = \Pi_1^\sharp \mathrm{d}F^* f + (F^* f) E_1 = \Pi_1^\sharp F^* \mathrm{d}f + (F^* f) E_1.$$

which further gives

$$TF(X_{F^* f}) = TF \circ \Pi_1^\sharp F^* \mathrm{d}f + (F^* f) TF(E_1).$$

Invoking Proposition 3.2.5, where the first relation in (3.54) is equivalent to $\Pi_2^\sharp = TF \circ \Pi_1^\sharp F^*$, it is clear that F is a Jacobi map if and only if (3.55) holds.

Remark 3.2.7. Assuming that $F : M_1 \to M_2$ is a Jacobi map, we analyze the relation between the pairs of forms (ϕ_1, ω_1) and (ϕ_2, ω_2). In view of this, we apply twice the Definition 3.2.4 and get

$$F^* \mathrm{Jac} \{f, g, h\}_2 = \mathrm{Jac} \{F^* f, F^* g, F^* h\}_1, \quad f, g, h \in \mathcal{F}(M_2).$$

The last result together with (3.46) and (3.55) further give

$$\begin{aligned}&\langle F^*\phi_2, X_{F^*f}\wedge X_{F^*g}\wedge X_{F^*h}\rangle - \left(\mathcal{L}_{E_1}(F^*f)\langle F^*\omega_2, X_{F^*g}\wedge X_{F^*h}\rangle + \text{cycl.}\right)\\ &= \langle \phi_1, X_{F^*f}\wedge X_{F^*g}\wedge X_{F^*h}\rangle - \left(\mathcal{L}_{E_1}(F^*f)\langle \omega_1, X_{F^*g}\wedge X_{F^*h}\rangle + \text{cycl.}\right),\end{aligned} \tag{3.56}$$

which eventually implies that

$$F^*\phi_2 = \phi_1, \quad F^*\omega_2 = \omega_1,$$

but only on the distribution generated by the Hamiltonian vector fields of the form $X_{F^*f} = \Pi_1^\sharp(F^*\mathrm{d}f) + (F^*f)E_1$ with $f \in \mathcal{F}(M_2)$, i.e. on the distribution $\Pi_1^\sharp((\ker TF)^\circ) + \langle E_1\rangle$.

Finally, by 'gluing' the bracket (3.50) to the concept of Jacobi map (see Definition 3.2.4) we are in the position to introduce the *conformal Jacobi map* as a smooth map F between the Jacobi manifolds with background $(M_i, ((\Pi_i, E_i), (\phi_i, \omega_i)))$, $i = 1, 2$ that fulfills

$$F^*\{f,g\}_2 = \{F^*f, F^*g\}_1^a, \quad f,g \in \mathcal{F}(M_2). \tag{3.57}$$

Definition (3.57) combined with the Propositions 3.2.3 and 3.2.5 ensure that the previous map is a conformal Jacobi one if and only if $\Pi_2 \circ F = TF \circ (a\Pi_1)$ and $E_2 \circ F = TF\left(aE_1 + \Pi_1^\sharp \mathrm{d}a\right)$. Moreover, because of (3.51), the background forms satisfy:

$$F^*\phi_2 = a^{-1}\phi_1 + \mathrm{d}\left(a^{-1}\right)\wedge\omega_1, \quad F^*\omega_2 = a^{-1}\omega_1$$

on the distribution $\Pi_1^\sharp((\ker TF)^\circ) + \langle X_a\rangle$ in M_1. These relations can be rewritten as

$$\phi_1 = aF^*\phi_2 + \mathrm{d}a\wedge F^*\omega_2, \quad \omega_1 = aF^*\omega_2. \tag{3.58}$$

3.3 "Poissonization" of Jacobi manifolds with background

In this section, we address the problem of a kind of Poissonization [76], [62] for a given Jacobi manifold with a background. In view of this, we define the *homogeneous Poisson manifold with background* as being a Poisson manifold with background $(M, (\Pi, \phi))$ that displays a vector field Z enjoying

$$[Z, \Pi] = -\Pi, \quad \mathcal{L}_Z\phi = \phi. \tag{3.59}$$

Proposition 3.3.1. *If $(M, (\Pi, E), (\phi, \omega))$ is a Jacobi manifold with background, then the manifold $\tilde{M} := M \times \mathbb{R}$ can be naturally endowed with a Poisson structure with background $(\tilde{\Pi}, \tilde{\phi})$,*

$$\tilde{\Pi} := \mathrm{e}^{-\tau}\left(\Pi + \partial_\tau \wedge E\right), \quad \tilde{\phi} := \mathrm{e}^{\tau}\left(\phi + \omega\wedge \mathrm{d}\tau\right), \tag{3.60}$$

that is homogeneous with respect to the vector field $Z := \partial_\tau$., i.e., (3.59) take place.

Proof. Putting together the relations (3.16) verified by the given Jacobi structure with background $((\Pi, E), (\phi, \omega))$ and the first definition in (3.60), by direct computations we get

$$\left[\tilde{\Pi}, \tilde{\Pi}\right] = 2\mathrm{e}^{-2\tau}\left[\Pi^\sharp\phi + \Pi^\sharp\omega\wedge E - \partial_\tau \wedge \left(\Pi^\sharp i_E\phi + \Pi^\sharp i_E\omega \wedge E\right)\right]. \tag{3.61}$$

Also, if we take $x = (x^i)$ a local chart on M such that

$$E = E^i \partial_i, \quad \omega = \frac{1}{2}\omega_{ij} \mathrm{d}x^i \wedge \mathrm{d}x^j, \quad \phi = \frac{1}{6}\phi_{ijk} \mathrm{d}x^i \wedge \mathrm{d}x^j \wedge \mathrm{d}x^k, \tag{3.62}$$

then the first definition in (3.60) gives

$$\tilde{\Pi}^\sharp \mathrm{d}x^i = \mathrm{e}^{-\tau}\left(\Pi^\sharp \mathrm{d}x^i - E^i \partial_\tau\right), \quad \tilde{\Pi}^\sharp \mathrm{d}\tau = \mathrm{e}^{-\tau} E. \tag{3.63}$$

At this stage, by means of the expressions (3.62) and (3.63) we obtain

$$\tilde{\Pi}^\sharp \phi = \mathrm{e}^{-3\tau}\left(\Pi^\sharp \phi - \partial_\tau \wedge \Pi^\sharp i_E \phi\right), \quad \tilde{\Pi}^\sharp \omega = \mathrm{e}^{-2\tau}\left(\Pi^\sharp \omega - \partial_\tau \wedge \Pi^\sharp i_E \omega\right). \tag{3.64}$$

Finally, inserting the results (3.64) into the right hand side of (3.61) and using the second definition in (3.60) we get that $\tilde{\Pi}$ and $\tilde{\phi}$ are subject to the equation (3.15). Moreover, direct computations lead to the fact that the geometric objects given in (3.60) verify (3.59) for $Z = \partial_\tau$. □

As it has been established in [62], by considering a background of the form $(\phi, \omega) = (\mathrm{d}\omega, \omega)$ in the previous proposition, we get an exact background 3-form $\tilde{\phi} = \mathrm{d}\left(\mathrm{e}^\tau \omega\right)$, hence the following result:

Corollary 3.3.2. *The "Poissonization" of a twisted Jacobi structure $((\Pi, E), \omega)$ on M is the manifold $\tilde{M} = M \times \mathbb{R}$, endowed with the twisted Poisson structure $(\tilde{\Pi}, \mathrm{d}(\mathrm{e}^\tau \omega))$.*

Remark 3.3.3. Let (θ, ω) be a twisted contact structure on the manifold M, with induced twisted Jacobi structure $((\Pi, E), \omega)$, offered by Proposition 3.1.12. Then, according to Proposition 3.3.1, the "Poissonization" yields the homogeneous Poisson structure with background $(\tilde{\Pi}, \tilde{\phi})$ on $M \times \mathbb{R}$, where $\tilde{\Pi}$ is non-degenerate with its inverse the 2-form

$$\tilde{\Omega} = e^\tau \left(\mathrm{d}\theta + \omega - \mathrm{d}\tau \wedge \theta\right) \tag{3.65}$$

and

$$\tilde{\phi} = \mathrm{d}\left(e^\tau \omega\right) = \mathrm{d}\tilde{\Omega}.$$

Thus we obtain a twisted symplectic structure $\tilde{\Omega}$ on $M \times \mathbb{R}$. Of course, when $\omega = 0$ this is nothing but the symplectization of a contact structure.

Proposition 3.3.4. *Let $(M_i, (\Pi_i, E_i), (\phi_i, \omega_i))$, $i = 1, 2$, be two Jacobi manifolds with background and $F : M_1 \to M_2$ be a smooth map. Then F is a Jacobi map if and only if the smooth map*

$$\tilde{F} : M_1 \times \mathbb{R} \to M_2 \times \mathbb{R}, \quad \tilde{F}(x, \tau) := (F(x), \tau)$$

is a Poisson map between the corresponding Poisson structures with background $(\tilde{\Pi}i, \tilde{\phi}_i)$, $i = 1, 2$.

Proof. Combining the results offered by the Propositions 3.2.5 and 3.3.1 with the expression of the tangent map $T\tilde{F}\left(X \oplus \tau\right) = TF(X) \oplus \tau$, we get $\tilde{\Pi}_2 \circ \tilde{F} = T\tilde{F} \circ \tilde{\Pi}_1$, and the conclusion follows. □

3.4 Transitive Jacobi manifolds with background

3.4.1 Characteristic distribution

The characteristic distribution corresponding to a Jacobi structure with background $((\Pi, E), (\phi, \omega))$ on a smooth manifold M is defined by

$$\mathcal{C}\,(M) = \bigcup_{p \in M} \mathcal{C}_p(M) := \bigcup_{p \in M} \langle \{E_p\} \cup \operatorname{Im} \Pi_p^\sharp \rangle. \tag{3.66}$$

It coincides with that generated by the Hamiltonian vector fields (3.44):

$$\mathcal{C}\,(M) = \bigcup_{p \in M} \left\langle \left\{ (X_f)_p : f \in \mathcal{F}(M) \right\} \right\rangle.$$

This further implies the smoothness of the characteristic distribution.

Proposition 3.4.1. *The characteristic distribution corresponding to a Jacobi structure with background is involutive.*

Proof. We show that

$$[X_f, X_g]\,(p) \in \mathcal{C}_p(M), \quad f, g \in \mathcal{F}(M).$$

Combining definitions (3.43) and (3.44), as well as Koszul relation (3.26), we successively obtain

$$\begin{aligned} [X_f, X_g]\,h - X_{\{f,g\}}h &= \operatorname{Jac}\{f,g,h\} + \{f, hEg\} - \{g, hEf\} + (X_g h)\,Ef \\ &\quad - (X_f h)\,Eg - hE\,\{f,g\} \\ &= \operatorname{Jac}\{f,g,h\} - h\,[\mathcal{L}_E, i_\Pi]\,(\mathrm{d}f \wedge \mathrm{d}g) \\ &= \operatorname{Jac}\{f,g,h\} - h i_{[E,\Pi]}\,(\mathrm{d}f \wedge \mathrm{d}g)\,. \end{aligned}$$

Inserting in the right hand side of the previous equality the relations (3.16) and (3.46), we get

$$[X_f, X_g] - X_{\{f,g\}} = \Pi^\sharp i_{X_f \wedge X_g}\phi - (Ef)\,\Pi^\sharp i_{X_g}\omega + (Eg)\,\Pi^\sharp i_{X_f}\omega + \left(i_{X_f \wedge X_g}\omega\right) E, \tag{3.67}$$

that proves the proposition. □

Definition 3.4.2. *A Jacobi manifold with background $(M, ((\Pi, E), (\phi, \omega)))$ is said to be transitive if its characteristic distribution $\mathcal{C}\,(M)$ coincides, at each point of the manifold, with the tangent space, i.e.*

$$T_pM = \left\langle \{E_p\} \cup \operatorname{Im} \Pi_p^\sharp \right\rangle, \quad p \in M. \tag{3.68}$$

Remark 3.4.3. From Definition 3.4.2 it is clear that in the context of transitive Jacobi structures with background, if M is even-dimensional then the 2-vector field Π is non-degenerate (i.e. $\Pi^\sharp$ is a vector bundle isomorphism with the inverse $\Pi^\flat$ [51]) and

$$E \in \operatorname{Im} \Pi^\sharp. \tag{3.69}$$

On the other hand, if M is odd-dimensional then $\dim \operatorname{Im} \Pi^\sharp + 1 = \dim M$ and

$$E_p \notin \operatorname{Im} \Pi_p^\sharp, \quad p \in M. \tag{3.70}$$

Example 3.4.4. Examples of transitive Jacobi structures with background are the locally conformal symplectic structures with background (see section 4.2.3) and the twisted contact structures (see section 3.1.4). We will see in Theorem 3.4.6 that these are the only transitive Jacobi structures with background.

3.4.2 Gauges for the background

Returning to Definition 3.1.3, it is natural to ask how much freedom is left for the exterior forms ϕ and ω, once we fix the vector and 2-vector fields E and Π respectively? We answer this question for transitive Jacobi manifolds with background as follows.

Theorem 3.4.5. *Let M be a smooth manifold and $((\Pi, E), (\phi_i, \omega_i))$, $i = 1, 2$ be two transitive Jacobi pairs with background on M. The following alternative holds:*

1. *If M is even-dimensional then there exists a 2-form ω, such that*

$$\omega_1 = \omega_2 + \omega, \quad \phi_1 = \phi_2 - \omega \wedge \Pi^\flat E; \tag{3.71}$$

2. *If M is odd-dimensional then*

$$\omega_1 = \omega_2, \quad \phi_1 = \phi_2. \tag{3.72}$$

Proof. When M is transitive even-dimensional, Remark 3.4.3 together with Definition 3.1.3 exhibit for the considered Jacobi structures with background

$$\tfrac{1}{2}[\Pi, \Pi] + E \wedge \Pi = \Pi^\sharp \phi_j + \Pi^\sharp \left(\omega_j \wedge \Pi^\flat E\right), \quad [E, \Pi] = -\left(\Pi^\sharp i_E \phi_j + \Pi^\sharp \left(i_E \omega_j \wedge \Pi^\flat E\right)\right), \tag{3.73}$$

for $j = 1, 2$. By subtraction we get

$$\Pi^\sharp \left(\phi + \omega \wedge \Pi^\flat E\right) = 0, \quad \Pi^\sharp i_E \left(\phi + \omega \wedge \Pi^\flat E\right) = 0, \tag{3.74}$$

where

$$\phi := \phi_1 - \phi_2, \quad \omega := \omega_1 - \omega_2. \tag{3.75}$$

Invoking the non-degeneracy of the 2-vector field Π, from (3.74) we infer $\phi = -\omega \wedge \Pi^\flat E$, hence (3.71).

In the transitive odd-dimensional situation, by using the same manipulations and the same notations (3.75) as in the even-dimensional case, we get

$$\Pi^\sharp \phi = -\Pi^\sharp \omega \wedge E, \quad \Pi^\sharp i_E \phi = -\Pi^\sharp i_E \omega \wedge E. \tag{3.76}$$

At this stage, in the light of Remark 3.4.3 (see (3.70)), relations (3.76) further impose

$$\Pi^\sharp i_E \omega = 0, \quad \Pi^\sharp \omega = 0, \quad \Pi^\sharp i_E \phi = 0, \quad \Pi^\sharp \phi = 0. \tag{3.77}$$

The first equation in (3.77) shows that $\operatorname{Im} \Pi^\sharp \subseteq \operatorname{Ker} i_E \omega$. This, combined with $E \in \operatorname{Ker} i_E \omega$, in the light of the transitiveness of the considered pairs (see (3.68)), allows us to conclude that $i_E \omega = 0$, which together with $\Pi^\sharp \omega = 0$, finally gives $\omega = 0$.

Now, by using the same arguments, we analyze the last two equations in (3.77)) and show that their unique solution is the trivial one $\phi = 0$. Let ρ and λ be two arbitrary 1-forms. With these objects at hand, the third equation in (3.77) is equivalent to $i_{\Pi^\sharp \rho}(i_{\Pi^\sharp \lambda} i_E \phi) = 0$. This, together with the obvious one $i_E(i_{\Pi^\sharp \lambda} i_E \phi) = 0$, by means of the transitiveness (3.68), exhibit $i_{\Pi^\sharp \lambda} i_E \phi = 0$ for all $\lambda \in \Omega^1(M)$, and, moreover $i_E \phi = 0$. Invoking again the transitive character of the considered Jacobi structures with background, the last result together with the fourth equation in (3.77) display $\phi = 0$, which completes the proof in the odd-dimensional context. □

The results given by the previous theorem offer the gauge transformations [70] for a given transitive Jacobi manifold with background, i.e. the changes of ω and ϕ that do not modify the fundamental objects Π and E respectively. In this sense the transitive odd-dimensional case is rigid. Thus for a twisted contact structure (θ, ω), which is known to be transitive odd-dimensional, the background $(\omega, \mathrm{d}\omega)$ cannot be altered.

3.4.3 Characterization of transitive Jacobi manifolds with background

The transitive twisted Poisson manifolds are twisted symplectic (alias almost symplectic) manifolds [68]. The transitive twisted Jacobi manifolds are either twisted contact or twisted locally conformal symplectic [64]. In the sequel, we shall characterize the transitive Jacobi manifolds with background.

Theorem 3.4.6. *Let $(M, ((\Pi, E), (\phi, \omega)))$ be a Jacobi manifold with background. If it is transitive then either it is a locally conformal symplectic manifold with background (see section 4.2.3) or it is a twisted contact one (see section 3.1.4).*

Proof. The argumentation takes into account the parity of the manifold M.

i) Assuming M is even-dimensional, from its transitivity we know that Π is non-degenerate, so we define the non-degenerate 2-form Ω as its inverse

$$\langle \Omega, X \wedge Y \rangle := \left\langle \Pi^\flat X \wedge \Pi^\flat Y, \Pi \right\rangle, \quad X, Y \in \mathfrak{X}^1(M), \tag{3.78}$$

and the 1-form α (the Lee 1-form) by

$$\alpha := -i_E \Omega. \tag{3.79}$$

At this stage we are in position to prove that the geometric objects defined in (3.79)–(3.78) verify the relations (3.24) and (3.25). In view of this we compute the de Rham differential of the 2-form Ω. Let f, g and h be three arbitrary smooth functions. By a direct computation based on definition (3.78) we get

$$\begin{aligned}
\left\langle \mathrm{d}\Omega, \Pi^\sharp \mathrm{d}f \wedge \Pi^\sharp \mathrm{d}g \wedge \Pi^\sharp \mathrm{d}h \right\rangle &= \sum_{\text{cyclic}} \left(\Pi^\sharp \mathrm{d}f \left(\left\langle \Omega, \Pi^\sharp \mathrm{d}g \wedge \Pi^\sharp \mathrm{d}h \right\rangle \right) - \left\langle \Omega, \left[\Pi^\sharp \mathrm{d}f, \Pi^\sharp \mathrm{d}g \right] \wedge \Pi^\sharp \mathrm{d}h \right\rangle \right) \\
&= i_\Pi \left(\mathrm{d}f \wedge \mathrm{d}i_\Pi \left(\mathrm{d}g \wedge \mathrm{d}h \right) \right) \\
&- \sum_{\text{cyclic}} \left(\left(\Pi^\sharp \mathrm{d}f \left(\Pi^\sharp \mathrm{d}g \left(h \right) \right) - \Pi^\sharp \mathrm{d}g \left(\Pi^\sharp \mathrm{d}f \left(h \right) \right) \right) \right)
\end{aligned}$$

$$
\begin{aligned}
&= -i_\Pi \mathrm{d} i_\Pi (\mathrm{d}f \wedge \mathrm{d}g \wedge \mathrm{d}h) \\
&- \sum_{\text{cyclic}} \left(\Pi^\sharp \mathrm{d}f \left(\left\langle \mathrm{d}h, \Pi^\sharp \mathrm{d}g \right\rangle\right) - \Pi^\sharp \mathrm{d}g \left(\left\langle \mathrm{d}h, \Pi^\sharp \mathrm{d}f \right\rangle\right)\right) \\
&= i_\Pi \mathrm{d} i_\Pi (\mathrm{d}f \wedge \mathrm{d}g \wedge \mathrm{d}h) = -\tfrac{1}{2} i_{[\Pi,\Pi]} (\mathrm{d}f \wedge \mathrm{d}g \wedge \mathrm{d}h) .
\end{aligned}
$$

In the same fashion, we successively obtain

$$
\begin{aligned}
\left\langle \alpha \wedge \Omega, \Pi^\sharp \mathrm{d}f \wedge \Pi^\sharp \mathrm{d}g \wedge \Pi^\sharp \mathrm{d}h \right\rangle &= \sum_{\text{cyclic}} \left\langle \alpha, \Pi^\sharp \mathrm{d}f \right\rangle \left\langle \Omega, \Pi^\sharp \mathrm{d}g \wedge \Pi^\sharp \mathrm{d}h \right\rangle \\
&= -\sum_{\text{cyclic}} \left\langle \mathrm{d}f, \Pi^\sharp \alpha \right\rangle i_\Pi (\mathrm{d}g \wedge \mathrm{d}h) \\
&= -\sum_{\text{cyclic}} i_E (\mathrm{d}f)\, i_\Pi (\mathrm{d}g \wedge \mathrm{d}h) = -i_{E\wedge\Pi} (\mathrm{d}f \wedge \mathrm{d}g \wedge \mathrm{d}h) .
\end{aligned}
$$

Adding the last two results and using the first relation in (3.16) we get

$$
\left\langle \mathrm{d}\Omega + \alpha \wedge \Omega, \Pi^\sharp \mathrm{d}f \wedge \Pi^\sharp \mathrm{d}g \wedge \Pi^\sharp \mathrm{d}h \right\rangle = -\left\langle \mathrm{d}f \wedge \mathrm{d}g \wedge \mathrm{d}h, \Pi^\sharp \phi + \Pi^\sharp \omega \wedge E \right\rangle ,
$$

or, equivalently

$$
\begin{aligned}
\left\langle \mathrm{d}\Omega + \alpha \wedge (\Omega - \omega), \Pi^\sharp \mathrm{d}f \wedge \Pi^\sharp \mathrm{d}g \wedge \Pi^\sharp \mathrm{d}h \right\rangle &= -\left\langle \mathrm{d}f \wedge \mathrm{d}g \wedge \mathrm{d}h, \Pi^\sharp \phi \right\rangle \\
&= \left\langle \phi, \Pi^\sharp (\mathrm{d}f \wedge \mathrm{d}g \wedge \mathrm{d}h) \right\rangle .
\end{aligned}
$$

which proves (3.25).

It remains to prove the closedness of the Lee 1-form α, (3.24). In view of this, we use its definition (3.79), relation (3.25) and the second equation in (3.16). By means of the first two arguments above we establish

$$
\mathrm{d}\alpha = -\mathrm{d} i_E \Omega = -\mathcal{L}_E \Omega + i_E \mathrm{d}\Omega = -\mathcal{L}_E \Omega + i_E \left[\phi - \alpha \wedge (\Omega - \omega)\right] . \tag{3.80}
$$

In the sequel, we compute the terms in the right hand side of (3.80). Let f and g be two arbitrary smooth functions. By using the relation between Lie derivatives and tensor contractions we get

$$
\begin{aligned}
\left\langle \mathcal{L}_E \Omega, \Pi^\sharp \mathrm{d}f \wedge \Pi^\sharp \mathrm{d}g \right\rangle &= \mathcal{L}_E \left\langle \Omega, \Pi^\sharp \mathrm{d}f \wedge \Pi^\sharp \mathrm{d}g \right\rangle - \left\langle \Omega, \left[E, \Pi^\sharp \mathrm{d}f\right] \wedge \Pi^\sharp \mathrm{d}g \right\rangle \\
&+ \left\langle \Omega, \left[E, \Pi^\sharp \mathrm{d}g\right] \wedge \Pi^\sharp \mathrm{d}f \right\rangle = -\mathcal{L}_E i_\Pi (\mathrm{d}f \wedge \mathrm{d}g) \\
&+ i_\Pi \mathcal{L}_E (\mathrm{d}f \wedge \mathrm{d}g) = -i_{[E,\Pi]} (\mathrm{d}f \wedge \mathrm{d}g) .
\end{aligned} \tag{3.81}
$$

Concerning the second term in the right hand side of (3.80), we subsequently obtain

$$
\left\langle i_E \phi, \Pi^\sharp \mathrm{d}f \wedge \Pi^\sharp \mathrm{d}g \right\rangle = \left\langle \mathrm{d}f \wedge \mathrm{d}g, \Pi^\sharp i_E \phi \right\rangle = i_{\Pi^\sharp i_E \phi} (\mathrm{d}f \wedge \mathrm{d}g) . \tag{3.82}
$$

Moreover, by means of the definition (3.79) combined with the antisymmetry of the 2-vector field Π we get $i_E \alpha = \Omega(E, E) = 0$, result that further leads to

$$
\left\langle i_E (\alpha \wedge \Omega), \Pi^\sharp \mathrm{d}f \wedge \Pi^\sharp \mathrm{d}g \right\rangle = -\left\langle \alpha \wedge i_E \Omega, \Pi^\sharp \mathrm{d}f \wedge \Pi^\sharp \mathrm{d}g \right\rangle = 0 \tag{3.83}
$$

and

$$\begin{aligned}\left\langle i_E\left(\alpha\wedge\omega\right),\Pi^\sharp \mathrm{d}f\wedge\Pi^\sharp \mathrm{d}g\right\rangle &= -\left\langle \alpha\wedge i_E\omega,\Pi^\sharp \mathrm{d}f\wedge\Pi^\sharp \mathrm{d}g\right\rangle\\ &= -\left\langle \alpha,\Pi^\sharp \mathrm{d}f\right\rangle\left\langle i_E\omega,\Pi^\sharp \mathrm{d}g\right\rangle + \langle\alpha,\Pi^\sharp \mathrm{d}g\rangle\langle i_E\omega,\Pi^\sharp \mathrm{d}f\rangle\\ &= -(Ef)\langle \mathrm{d}g,\Pi^\sharp i_E\omega\rangle + (Eg)\langle \mathrm{d}f,\Pi^\sharp i_E\omega\rangle\\ &= i_{\Pi^\sharp i_E\omega\wedge E}\left(\mathrm{d}f\wedge\mathrm{d}g\right)\end{aligned} \tag{3.84}$$

Inserting the relations (3.81), (3.82), (3.83) and (3.84) in the right hand side of the formula (3.80) and invoking Definition 3.1.3 (see the second equation in (3.16)) we prove that $\mathrm{d}\alpha = 0$.

ii) We analyze the transitive Jacobi structures with background over odd-dimensional manifolds. In this context, we prove that the considered Jacobi structure is a twisted contact structure (see Section 3.1.4). In view of this, we start from Remark 3.4.3 and construct the 1-form θ such that

$$i_E\theta = 1, \quad i_{\Pi^\sharp\lambda}\theta = 0, \quad \lambda\in\Omega^1(M), \tag{3.85}$$

where the second identity means $\Pi^\sharp\theta = 0$. In order to show that (θ,ω) is a twisted contact structure over the base manifold M, it is enough to prove that the 2-form $\mathrm{d}\theta+\omega$ verifies the equations

$$i_E(\mathrm{d}\theta+\omega) = 0, \quad i_{\Pi^\sharp\lambda}(\mathrm{d}\theta+\omega) = i_E\left(\lambda\wedge\theta\right), \quad \lambda\in\Omega^1\left(M\right). \tag{3.86}$$

Indeed, due to the fact that E and $\Pi^\sharp\lambda$ generate the whole tangent space, the relations (3.85) and (3.86) imply the non-degeneracy of the volume form (3.34) as

$$i_E\mu = -(\mathrm{d}\theta+\omega)^m, \quad i_{\Pi^\sharp\lambda}\mu = -m\lambda\wedge\theta\wedge(\mathrm{d}\theta+\omega)^{m-1}, \quad \mathrm{dim}M = 2m+1. \tag{3.87}$$

We establish the first relation in (3.86) by computing $i_E(\mathrm{d}\theta+\omega)$. Let λ be an arbitrary 1-form, $\lambda\in\Omega^1(M)$. Starting with relations (3.85), we successively derive

$$\begin{aligned}i_{\Pi^\sharp\lambda}i_E(\mathrm{d}\theta+\omega) = i_{E\wedge\Pi^\sharp\lambda}(\mathrm{d}\theta+\omega) = i_{E\wedge\Pi^\sharp\lambda}\left(\mathrm{d}\theta+\omega\right) &= \left\langle\omega,E\wedge\Pi^\sharp\lambda\right\rangle + \left\langle \mathrm{d}\theta,E\wedge\Pi^\sharp\lambda\right\rangle\\ = -\left\langle\lambda,\Pi^\sharp i_E\omega\right\rangle\left\langle\theta,E\right\rangle + \left\langle \mathrm{d}\theta,E\wedge\Pi^\sharp\lambda\right\rangle&\\ = -\left\langle\lambda\wedge\theta,\Pi^\sharp i_E\omega\wedge E\right\rangle + \left\langle \mathrm{d}\theta,E\wedge\Pi^\sharp\lambda\right\rangle.&\end{aligned} \tag{3.88}$$

Invoking the second equation in (3.16) and the second relation in (3.85), direct computations based on Koszul relation (3.26) further give

$$\begin{aligned}-\left\langle\lambda\wedge\theta,\Pi^\sharp i_E\omega\wedge E\right\rangle &= \left\langle\lambda\wedge\theta,[E,\Pi]+\Pi^\sharp i_E\phi\right\rangle = \langle\lambda\wedge\theta,[E,\Pi]\rangle\\ &= -\left[\left[i_\Pi,\mathrm{d}\right],i_E\right]\left(\lambda\wedge\theta\right) = i_\Pi i_E\left(\lambda\wedge \mathrm{d}\theta\right) - i_\Pi \mathrm{d}\left(\left(i_E\lambda\right)\theta\right)\\ &= -i_{\Pi^\sharp\lambda}i_E\mathrm{d}\theta = -\left\langle \mathrm{d}\theta,E\wedge\Pi^\sharp\lambda\right\rangle.\end{aligned} \tag{3.89}$$

Putting together (3.88) and (3.89) we conclude that

$$i_{\Pi^\sharp\lambda}i_E(\mathrm{d}\theta+\omega) = 0, \tag{3.90}$$

which, supplemented with $i_Ei_E(\mathrm{d}\theta+\omega) = 0$, prove the first relation in (3.86).

Using the same reasoning, we prove the second formula in (3.86). Firstly, by means of the first relation in (3.86), we immediately deduce

$$i_Ei_{\Pi^\sharp\lambda}(\mathrm{d}\theta+\omega) = -i_{\Pi^\sharp\lambda}i_E(\mathrm{d}\theta+\omega) = 0 = i_Ei_E\left(\lambda\wedge\theta\right), \quad \lambda\in\Omega^1(M). \tag{3.91}$$

Next, we consider two arbitrary 1-forms ρ and λ and show that the second formula in (3.86) also holds on the pair $\left(\Pi^\sharp\rho, \Pi^\sharp\lambda\right)$. Direct computations based on (3.85) display

$$\begin{aligned}
i_{\Pi^\sharp\rho} i_{\Pi^\sharp\lambda}(\mathrm{d}\theta+\omega) &= \langle \mathrm{d}\theta+\omega, \Pi^\sharp\lambda\wedge\Pi^\sharp\rho\rangle = \langle \mathrm{d}\theta, \Pi^\sharp\lambda\wedge\Pi^\sharp\rho\rangle + \langle \lambda\wedge\rho, \Pi^\sharp\omega\rangle \\
&= \langle \mathrm{d}\theta, \Pi^\sharp\lambda\wedge\Pi^\sharp\rho\rangle + \langle \lambda\wedge\rho, \Pi^\sharp\omega\rangle\langle\theta, E\rangle \\
&= \langle \mathrm{d}\theta, \Pi^\sharp\lambda\wedge\Pi^\sharp\rho\rangle + \langle \lambda\wedge\rho\wedge\theta, \Pi^\sharp\omega\wedge E\rangle \\
&= \langle \mathrm{d}\theta, \Pi^\sharp\lambda\wedge\Pi^\sharp\rho\rangle + \langle \lambda\wedge\rho\wedge\theta, \tfrac{1}{2}\left[\Pi,\Pi\right] + E\wedge\Pi - \Pi^\sharp\phi\rangle \\
&= \langle \mathrm{d}\theta, \Pi^\sharp\lambda\wedge\Pi^\sharp\rho\rangle + \tfrac{1}{2}\langle \lambda\wedge\rho\wedge\theta, [\Pi,\Pi]\rangle + \langle\lambda\wedge\rho, \Pi\rangle \\
&= \langle \mathrm{d}\theta, \Pi^\sharp\lambda\wedge\Pi^\sharp\rho\rangle + \tfrac{1}{2} i_{[\Pi,\Pi]}\left(\lambda\wedge\rho\wedge\theta\right) + i_\Pi\left(\lambda\wedge\rho\right). \qquad (3.92)
\end{aligned}$$

The right hand side of (3.92) can be simplified via the Gelfand and Dorfman formula [29]:

$$\tfrac{1}{2} i_{[\Pi,\Pi]}\left(\lambda\wedge\rho\wedge\theta\right) = \langle\theta, \left[\Pi^\sharp\lambda, \Pi^\sharp\rho\right]\rangle - \langle\theta, \Pi^\sharp\left[\lambda,\rho\right]_\Pi\rangle, \qquad (3.93)$$

for the Dorfman bracket

$$\left[\lambda,\rho\right]_\Pi := \mathcal{L}_{\Pi^\sharp\lambda}\rho - \mathcal{L}_{\Pi^\sharp\rho}\lambda - d i_\Pi\left(\lambda\wedge\rho\right).$$

Indeed, by using (3.93) together with the second equation in (3.85), which is equivalent to $\Pi^\sharp\theta = 0$, we get

$$\tfrac{1}{2} i_{[\Pi,\Pi]}\left(\lambda\wedge\rho\wedge\theta\right) = \langle\theta, \left[\Pi^\sharp\lambda, \Pi^\sharp\rho\right]\rangle = -\langle\mathrm{d}\theta, \Pi^\sharp\lambda\wedge\Pi^\sharp\rho\rangle. \qquad (3.94)$$

Inserting the result (3.94) into the right hand side of the equality (3.92) we obtain

$$i_{\Pi^\sharp\rho} i_{\Pi^\sharp\lambda}(\mathrm{d}\theta+\omega) = i_\Pi\left(\lambda\wedge\rho\right) \qquad (3.95)$$

Finally, equations (3.85) further imply that

$$i_{\Pi^\sharp\rho} i_E\left(\lambda\wedge\theta\right) = -i_E i_{\Pi^\sharp\rho}\left(\lambda\wedge\theta\right) = -i_E\left(\left(i_{\Pi^\sharp\rho}\lambda\right)\theta\right) = -i_{\Pi^\sharp\rho}\lambda = i_\Pi\left(\lambda\wedge\rho\right),$$

which, in the light of (3.95), allows to show that

$$i_{\Pi^\sharp\rho} i_{\Pi^\sharp\lambda}(\mathrm{d}\theta+\omega) = i_{\Pi^\sharp\rho} i_E\left(\lambda\wedge\theta\right). \qquad (3.96)$$

Putting together the results (3.91) and (3.96) we conclude that the second equation in (3.86) also holds. □

At this point we have all the tools to formulate the main result of this section.

Theorem 3.4.7. *The characteristic distribution associated to a Jacobi manifold with background is completely integrable (in the sense of Stefan-Sussmann). Its characteristic leaves are either locally conformal symplectic manifolds with background (if even dimensional) or twisted contact manifolds (if odd dimensional).*

Proof. The proof uses the involutivity of characteristic distribution (see Proposition 3.4.1) combined with the fact that characteristic distribution possesses constant rank along the flow lines of its sections (this is done in [42, Theorem 1]) and supplemented with the fact that through each point of the smooth manifold M only one maximal integral submanifold passes (this can be found in [64, Theorem 3.2]). □

Corollary 3.4.8. *The pull-back of the background 3-form ϕ to an odd dimensional characteristic leaf is always exact.*

Proof. Let $i : L \hookrightarrow M$ be an odd-dimensional characteristic leaf, thus twisted contact (by Theorem 3.4.7). In particular its background $(i^*\phi, i^*\omega)$ has to satisfy $i^*\phi = \mathrm{d}i^*\omega$, so it is exact. □

3.5 Twisted dual pairs

The twisted version of a (symplectic) dual pair [82] has been studied in [37]. In this section we give a twisted version (in the trivial line bundle setting) of a contact dual pair [73].

3.5.1 Twisted symplectic dual pairs

Let (M, Ω) be a twisted symplectic manifold (with 3-form $\phi = \mathrm{d}\Omega$), and let (P_1, Π_1, ϕ_1), (P_2, Π_2, ϕ_2) be two Poisson manifolds with background. A pair of Poisson maps

$$\begin{array}{ccc} & (M,\Omega) & \\ {\scriptstyle F_1}\swarrow & & \searrow{\scriptstyle F_2} \\ (P_1,\Pi_1,\phi_1) & & (P_2,\Pi_2,\phi_2) \end{array} \tag{3.97}$$

is called a *twisted symplectic dual pair* [37] if $\ker TF_1$ and $\ker TF_2$ are orthogonal complements of each other, with respect to the non-degenerate 2-form Ω.

The twisted symplectic groupoids [12] are examples of twisted symplectic dual pairs.

Remark 3.5.1. For all $f_1 \in \mathcal{F}(P_1)$ and $f_2 \in \mathcal{F}(P_2)$, we get commuting functions

$$\{F_1^* f_1, F_2^* f_2\} = 0$$

for the bracket on $\mathcal{F}(M)$ induced by Ω. Indeed, the bracket can be written as

$$\{F_1^* f_1, F_2^* f_2\} = i_{X_{F_1^* f_1}} \mathrm{d}(F_2^* f_2),$$

but the Hamiltonian vector fields $X_{F_1^* f_1}$ generate $(\ker TF_1)^\perp = \ker TF_2$, hence the vanishing of the bracket.

Remark 3.5.2. Because of the relation (3.46) between the Jacobiator and the background form, as in Remark 3.2.7 we obtain that the 3-forms $F_i^*\phi_i$ and $\phi = \mathrm{d}\Omega$ coincide on $\Pi^\sharp((\ker TF_i)^\circ) = (\ker TF_i)^\perp$, for $i = 1, 2$. From this, with the dual pair condition $(\ker TF_1)^\perp = \ker TF_2$, we deduce a relation between the 3-forms on the three manifolds, namely

$$\phi = \mathrm{d}\Omega = F_1^*\phi_1 + F_2^*\phi_2$$

on the integrable distribution

$$\ker TF_1 + \ker TF_2 = \ker TF_i + (\ker TF_i)^\perp, \quad i = 1, 2.$$

Notice that the background 3-forms ϕ_1, ϕ_2 need not be exact or closed.

Proposition 3.5.3. *[37] If the Poisson maps in the twisted symplectic dual pair* (3.97) *are surjective submersions (i.e. the dual pair is full) with connected fibers, then there is a one-to-one correspondence between the characteristic (twisted symplectic) leaves of* P_1 *and* P_2*, namely* $L_2 = F_2(F_1^{-1}(L_1))$ *with inverse* $L_1 = F_1(F_2^{-1}(L_2))$*.*

3.5.2 Twisted contact dual pairs

We consider a twisted contact manifold (M, θ, ω), with non-degenerate 2-form $\Omega = \mathrm{d}\theta + \omega$ on the hyperplane distribution $\mathcal{H} = \ker\theta$, as well as two Jacobi manifolds with background, $(P_1, (\Pi_1, E_1), (\phi_1, \omega_1))$ and $(P_2, (\Pi_2, E_2), (\phi_2, \omega_2))$.

Definition 3.5.4. A pair of conformal Jacobi maps with conformal factors $a_1, a_2 \in \mathcal{F}(M)$,

$$\begin{array}{ccc} & (M,\theta,\omega) & \\ {}^{F_1}\swarrow & & \searrow^{F_2} \\ P_1 & & P_2 \end{array} \tag{3.98}$$

is called a *twisted contact dual pair* if the following conditions hold:

1. $\mathcal{H}$ is transverse to both $\ker TF_1$ and $\ker TF_2$;
2. $\{a_1, a_2\} = 0$ and $X_{a_1} \in \Gamma(\ker TF_2)$, $X_{a_2} \in \Gamma(\ker TF_1)$;
3. the F_1-vertical part $\mathcal{H}_1 := \ker TF_1 \cap \mathcal{H}$ and the F_2-vertical part $\mathcal{H}_2 := \ker TF_2 \cap \mathcal{H}$ of $\mathcal{H}$ are orthogonal complements of each other, i.e. $\mathcal{H}_1^\perp = \mathcal{H}_2$ with respect to the non-degenerate 2-form $\Omega = \mathrm{d}\theta + \omega$ on $\mathcal{H}$.

The twisted contact dual pair is called *full* if the maps F_1 and F_2 are surjective submersions.

The twisted contact groupoids [65] are examples of full twisted contact dual pairs.

Remark 3.5.5. From the identity $\Pi^\sharp((\ker TF_1)^\circ) = \mathcal{H}_1^\perp = \mathcal{H}_2$ that follows from (3.39) and 3, we deduce that $\Pi(\mathrm{d}F_1^* f_1, \mathrm{d}F_2^* f_2) = 0$ for all $f_1 \in \mathcal{F}(P_1)$ and $f_2 \in \mathcal{F}(P_2)$. Using also point 2 in the definition of a twisted contact dual pair, we conclude that

$$\{a_1 F_1^* f_1, a_2 F_2^* f_2\} = 0, \quad f_1 \in \mathcal{F}(P_1), f_2 \in \mathcal{F}(P_2).$$

Given a surjective submersion $F : M \to P$ with connected fibers, the *pullback of a distribution* $\mathcal{C} \subseteq TP$ is the distribution $F^*\mathcal{C} \subseteq TM$ defined at each $p \in M$ by $(F^*\mathcal{C})_p := (T_pF)^{-1}(\mathcal{C}_{F(p)})$. It is an integrable distribution, provided the distribution $\mathcal{C}$ is integrable. From Appendix E in [10] we know that, with the additional assumption that the fibers are connected, there is a one-to-one correspondence between leaves L of $\mathcal{C}$ and K of $F^*\mathcal{C}$, given by

$$L \mapsto K = F^{-1}(L) \text{ and } K \mapsto L = F(K). \tag{3.99}$$

Lemma 3.5.6. *In a full twisted contact dual pair, the pullback of the characteristic distributions* $\mathcal{C}_1$ *and* $\mathcal{C}_2$ *on the Jacobi manifolds* P_1 *and* P_2 *coincide. Moreover,*

$$F_1^*\mathcal{C}_1 = \ker TF_1 + \ker TF_2 = F_2^*\mathcal{C}_2. \tag{3.100}$$

Proof. We compute the kernel of TF_2 as

$$\ker TF_2 \overset{2.}{=} \mathcal{H}_2 + \langle X_{a_1} \rangle \overset{3.}{=} \mathcal{H}_1^\perp + \langle X_{a_1} \rangle \overset{(3.39)}{=} \Pi^\sharp((\ker TF_1)^\circ) + \langle X_{a_1} \rangle. \tag{3.101}$$

Its image under TF_1 is the characteristic distribution $\mathcal{C}_1 = \operatorname{Im}\Pi_1 + \langle E_2 \rangle$, because by Proposition 3.2.5 the vector fields X_{a_1} and E_1, as well as the bi-vector fields $a_1\Pi$ and Π_1, are related by the map F_1 which is conformal Jacobi with conformal factor a_1. Thus $\ker TF_1 + \ker TF_2 = F_1^*\mathcal{C}_1$. □

Remark 3.5.7. From (3.58) we deduce that $\mathrm{d}\omega = a_1 F_1^*\phi_1 + \mathrm{d}a_1 \wedge F_1^*\omega_1$ and $\omega = a_1 F_1^*\omega_1$ on the distribution

$$\Pi^\sharp((\ker TF_1)^\circ) + \langle X_{a_1} \rangle \overset{(3.101)}{=} \ker TF_2,$$

and similarly for the complementary indices. Thus, we get relations between the twisting forms, namely

$$\mathrm{d}\omega = a_1 F_1^*\phi_1 + \mathrm{d}a_1 \wedge F_1^*\omega_1 + a_2 F_2^*\phi_2 + \mathrm{d}a_2 \wedge F_2^*\omega_2, \quad \omega = a_1 F_1^*\omega_1 + a_2 F_2^*\omega_2,$$

when restricted to the integrable distribution $\ker TF_1 + \ker TF_2$.

Proposition 3.5.8. *If the Poisson maps in the full twisted contact dual pair* (3.98) *have connected fibers, then there is a one-to-one correspondence between the characteristic leaves of P_1 and P_2, namely $L_2 = F_2(F_1^{-1}(L_1))$ with inverse $L_1 = F_1(F_2^{-1}(L_2))$.*

Moreover, the odd dimensional (twisted contact) leaves are in correspondence to each other, as well as the even dimensional (locally conformal symplectic with background) leaves.

The proof uses the fact that for full dual pairs (3.98) with $\dim M = 2m+1$,

$$\dim P_1 + \dim P_2 = 2m. \tag{3.102}$$

Indeed, let $k_1 := \dim\ker TF_1$ and $k_2 := \dim\ker TF_2$. Thus $\dim\mathcal{H}_1 = k_1 - 1$ and $\dim\mathcal{H}_2 = k_2 - 1$ by the transversality conditions at point 1 in the contact dual pair definition. From $\mathcal{H}_1^\perp = \mathcal{H}_2$ we get $k_1 + k_2 = 2m+2$, hence $\dim P_1 + \dim P_2 = 2m+1-k_1+2m+1-k_2 = 2m$.

Proof. We know there is a one-to-one correspondence between leaves of $\mathcal{C}_i$ and $F_i^*\mathcal{C}_i$ given by (3.99): $L_i \mapsto F_i^{-1}(L_i)$ and $K \mapsto F_i(K)$, for $i = 1,2$. Thus by (3.100) there is a one-to-one correspondence between characteristic leaves of P_1 and P_2, namely $L_2 = F_2(F_1^{-1}(L_1))$ with inverse $L_1 = F_1(F_2^{-1}(L_2))$.

Let $d = \dim(\ker TF_1 \cap \ker TF_2)$, so that $\dim(\ker TF_1 + \ker TF_2) = k_1 + k_2 - d = 2m+2-d$, because of (3.102). Counting the dimensions in (3.100) one gets that $k_1 + \dim L_1 = 2m + 2 - d = k_2 + \dim L_2$, thus $\operatorname{codim} L_1 = 2m+1-k_1-\dim L_1 = d-1$ is the same as $\operatorname{codim} L_2$. But by (3.102) the dimensions of P_1 and P_2 have the same parity. Thus the dimensions of the leaves L_1 and L_2 also have the same parity. □

The "Poissonization" procedure from Section 3.3 can be applied to all the objects involved in a twisted contact dual pair.

Proposition 3.5.9. *Given a twisted contact dual pair* (3.98)*, the pair of Poisson maps*

$$\begin{array}{ccc} & (M\times\mathbb{R},\tilde{\Omega}) & \\ \tilde{F}_1 \swarrow & & \searrow \tilde{F}_2 \\ (P_1\times\mathbb{R},\tilde{\Pi}_1,\tilde{\phi}_1) & & (P_2\times\mathbb{R},\tilde{\Pi}_2,\tilde{\phi}_2) \end{array} \tag{3.103}$$

forms a homogeneous twisted symplectic dual pair.

Proof. We verify that the twisted symplectic dual pair condition

$$\ker T\tilde{F}_1 = (\ker T\tilde{F}_2)^{\perp}$$

with respect to the non-degenerate 2-form $\tilde{\Omega} = e^{\tau}(\mathrm{d}\theta + \omega - \mathrm{d}\tau \wedge \theta)$ obtained by the "symplectization" of the twisted contact structure (θ, ω) (3.65). First we notice that the kernel of the Poisson map $\tilde{F}_i$ is

$$\ker T\tilde{F}_i = \{(-a_i^{-1} da_i(X), X) | X \in \ker TF_i\} \tag{3.104}$$

For the inclusion $\ker T\tilde{F}_1 \subseteq (\ker T\tilde{F}_2)^{\perp}$, we choose arbitrary vectors $X \in \ker TF_1$ and $Y \in \ker TF_2$. Let us denote $H_X = X - a_2^{-1}\theta(X)X_{a_2} \in \mathcal{H}_1$ and $H_Y = Y - a_1^{-1}\theta(Y)X_{a_1} \in \mathcal{H}_2$, and we compute:

$$\begin{aligned}
\tilde{\Omega}((-a_1^{-1}da_1(X), X), (-a_2^{-1}da_2(Y), Y)) &\overset{(3.65)}{=} (\mathrm{d}\theta + \omega)(X, Y) + \theta(X)a_2^{-1}\mathrm{d}a_2(Y) \\
&- \theta(Y)a_1^{-1}\mathrm{d}a_1(X) = (\mathrm{d}\theta + \omega)(H_X, H_Y) \\
&+ a_1^{-1}a_2^{-1}\theta(X)\theta(Y)(\mathrm{d}\theta + \omega)(X_{a_2}, X_{a_1}) \\
&+ a_2^{-1}\theta(X)((\mathrm{d}\theta + \omega)(X_{a_2}, H_Y) + \mathrm{d}a_2(Y)) \\
&- a_1^{-1}\theta(Y)((\mathrm{d}\theta + \omega)(X_{a_1}, H_X) + \mathrm{d}a_1(X)) \\
&\overset{(3.37)}{=} (\mathrm{d}\theta + \omega)(H_X, H_Y) + a_1^{-1}a_2^{-1}\theta(X)\theta(Y)(-\Pi(\mathrm{d}a_1, \mathrm{d}a_2) \\
&+ \mathrm{d}a_2(X_{a_1}) - \mathrm{d}a_1(X_{a_2})) \overset{(3.44)}{=} (\mathrm{d}\theta + \omega)(H_X, H_Y) \\
&+ a_1^{-1}a_2^{-1}\theta(X)\theta(Y)(\Pi(\mathrm{d}a_1, \mathrm{d}a_2) + i_E(a_1\mathrm{d}a_2 - a_2\mathrm{d}a_1)) \\
&\overset{(3.43)}{=} (\mathrm{d}\theta + \omega)(H_X, H_Y) + a_1^{-1}a_2^{-1}\theta(X)\theta(Y)\{a_1, a_2\} = 0,
\end{aligned}$$

expression that vanishes because of points 2 and 3 in the twisted contact dual pair definition. The reverse inclusion follows from a dimension count:

$$\dim\ker T\tilde{F}_1 + \dim\ker T\tilde{F}_2 = \dim\ker TF_1 + \dim\ker TF_2 = k_1 + k_2 = 2m + 2 = \dim(M \times \mathbb{R}).$$

□

Chapter 4

Jacobi bundles with background

In the early '70s it was invented a new geometric concept [42] that puts on equal footing locally conformal symplectic and contact structures. This assumes a vector bundle, $(E \to M)$ and a *local* Lie algebra structure over the $\mathbb{R}$-vector space of smooth sections, $\Gamma(E)$. In appropriate circumstances (precisely, when the vector bundle reduces to the trivial line bundle $\mathbb{R}_M \to M$), this geometric structure gives rise the well-known Jacobi manifold [33, 46, 24]. Globally, the Jacobi manifold was initially defined via a pair (lately addressed [73] as the Jacobi pair) consisting of a 2-vector and a vector field subject to two consistency conditions that make use of the Schouten bracket associated with the Gerstenhaber a1gebra of multi-vector fields. These conditions can be reformulated in terms of a Maurer-Cartan equation [62, 73] related to the Gerstenhaber-Jacobi algebra structure over the set of the first-order multi-derivations of the trivial line bundle, which shows that Jacobi pair generalizes in some sense Poisson structure. Also, the Jacobi pair organizes the $\mathbb{R}$-vector space of smooth functions as a Lie algebra (with respect to the well-known Jacobi bracket) but not a Poisson one.

It is worth mentioning that the previous structure, via the associated Jacobi bracket, has recently found many applications in mathematical physics, namely in the canonical approach of non-autonomous Hamiltonian systems [81, 76], in the integrability of Hamiltonian systems on odd-dimensional manifolds [77, 78, 45], in the construction of Jacobi Sigma Models [4] as well as in the geometric reformulation of non-equilibrium thermodynamics [5].

A straightforward generalization of the Jacobi pair comes from its 'twist' [62] (at the level of the Jacobi identity for the associated Jacobi bracket) by a 2-form and its de Rham differential. Twisted Jacobi manifolds (manifolds equipped with twisted Jacobi pairs) enjoy the main features of Jacobi manifolds: i) their characteristic distributions are completely integrable, with twisted cooriented contact structures on the odd-dimensional leaves and twisted locally conformal symplectic structures on the even-dimensional leaves [64] and ii) they are in one-to-one correspondence with homogeneous twisted Poisson manifolds, where the background 3-form is exact [62].

In the previous chapter, we showed that the twisted Jacobi pair concept is encompassed by the Jacobi *pair with background*. As for any Jacobi-like pair, this starts with a pair consisting of a 2-vector and a vector field and adds a 'background' (that spoils the Jacobi identity for the Jacobi bracket) comprising a 3-form together with a 2-form. If in a Jacobi pair with background, the 3-form reduces to the de Rham differential of the 2-form, then this reduces to a twisted Jacobi pair. In addition, Jacobi manifolds with background (those equipped with Jacobi pairs with background) enjoys the main features of the Jacobi and twisted Jacobi manifolds, i.e. i) their characteristic distributions are completely integrable, with twisted cooriented contact structures on the odd-dimensional leaves and locally conformal symplectic structures with background on the even-dimensional leaves [16] and ii) they are in one-to-one correspondence with homogeneous Poisson manifolds with background, where the background 3-form is no longer closed [16]. At this point, it is worth noticing the very recent interest in physics for Poisson Sigma Models with non-closed background 3-form and their relation with twisted Jacobi Sigma Models [13].

Recently, employing the Gerstenhaber-Jacobi algebra structure of the multi-derivations of the trivial line bundle, it has been shown that the consistency conditions corresponding to various Jacobi-like pairs [73, 62, 16] can be compactly written as Maurer-Cartan-like equations. These results together with the algebraic characterisations of Lie and Jacobi algebroids [31, 32] allow the introduction of the line-bundle versions encompassing the previous 'pairs'. Within this global setting, the 'pairs' are nothing but the trivial line-bundle versions of the corresponding Jacobi-like bundles [56, 73]. Starting from this remark, the aim of this chapter consists of the analysis of twisted Jacobi and Jacobi bundles with background. The analysis will follow the strategy in [73, 79] and will be mainly focused on the integrability of the characteristic distributions corresponding to twisted Jacobi and Jacobi bundles with background.

The present chapter is organized into four sections as follows. Section 1 is dedicated to standard characterizations [31, 32] of Lie and Jacobi algebroids. Unavoidable, this includes the Atiyah algebroid of the derivations of a line bundle [43]. For connecting the line-bundle formulations of the analyzed Jacobi bundles with the 'pairs', the previously mentioned results are done also in the trivial line bundle context [73]. For self-consistency reasons, in Section 2 we shortly address the Jacobi bundles [56] and their characteristic distributions integrability [23, 73]. Section 3 is dedicated to twisted Jacobi bundles. In the literature, this has been previously done only in the context of the trivial line bundle, i.e., in our language, twisted Jacobi pairs. Here, we collect the main results concerning transitive twisted Jacobi bundles and the integrability of twisted Jacobi bundles. Section 4 is dedicated to the main aim of the present chapter, namely Jacobi bundles with background. With the preparations made in Secs. 1–3 of the present chapter, it is shown that the trivial line bundle version of a Jacobi bundle with background is nothing but a Jacobi pair with background. Then, the analysis of transitive Jacobi bundles with background allows us to conclude that they are equivalent to either a locally conformal symplectic structure with background or a twisted conformal structure. Finally, by using the fact that locally, any Jacobi bundle with background is equivalent to a Jacobi pair with background, we sketch the proof of integrability of the characteristic distribution associated with a Jacobi bundles with background.

The original results contained in the present chapter are based on [16, 19, 20].

4.1 Lie and Jacobi algebroids: A brief review

Since the 'birth' of the local Lie algebras [42], it has been clear that they should be somehow connected with already known Lie algebroids [66]. This connection is pointed out by the bracket among sections in the vector bundle enjoying the locality (i.e. the bracket does not increase the support of its factors). In fact, Lie algebroids, introduced in the 1960s by Pradines [66], represent the natural framework for Jacobi and Jacobi-like bundles. In this section, we initially collect some results [31, 32] concerning the algebraic characterization of Lie algebroids through either a Gerstenhaber algebra or a differential complex (de Rham). Then, with the help of a derivative representation (on a line bundle) [43] of a given Lie algebroid, we implement and algebraically reformulate [31, 32, 73] the concept of Jacobi algebroid. Due to the fact that we are to prove that Jacobi-like pairs are nothing but Jacobi-like trivial line bundles, we focus a little bit on Jacobi algebroids over trivial line bundles.

By its very definition, a Lie algebroid is a vector bundle $A \to M$ equipped with a Lie algebra structure on the set of its smooth sections, $\Gamma(A)$, which in turn, is endowed with a vector bundle map (the well-known anchor), $\rho : A \to TM$, that is a Lie algebra morphism

$$\rho\left(\left[\alpha,\beta\right]\right) = \left[\rho\left(\alpha\right),\rho\left(\beta\right)\right], \quad \alpha,\beta \in \Gamma\left(A\right), \tag{4.1}$$

which is a first-order differential operator in each factor

$$\left[\alpha, f\beta\right] = \left(\rho\left(\alpha\right) f\right)\beta + f\left[\alpha,\beta\right], \quad \alpha,\beta \in \Gamma\left(A\right), f \in \mathcal{F}\left(M\right). \tag{4.2}$$

As we have specified in the beginning of this work, we denote by $\mathcal{F}(M)$ the associative, commutative, and unital algebra of real smooth functions on the smooth manifold M. A given Lie algebroid structure on $A \to M$ can be agebraically reformulated [31, 32, 73] as follows.

Theorem 4.1.1. *Let $A \to M$ be a vector bundle. Then, the following data are equivalent:*

1. *a Lie algebroid structure, $([\bullet,\bullet],\rho)$, on $A \to M$;*
2. *a Gerstenhaber algebra structure, $[\bullet,\bullet]_A$, on the graded algebra $\mathcal{A}^\bullet_A := \Gamma\left(\wedge^\bullet A\right)$;*
3. *a homological degree 1 graded derivation, d_A, acting on the graded algebra $\tilde{\mathcal{A}}^\bullet_A := \Gamma\left(\wedge^\bullet A^*\right)$.*

Because the graded algebra $\mathcal{A}^\bullet_A$ is generated by its zeroth and first-degree components, it results that the one-to-one correspondence between Gerstenhaber and Lie algebroid structures reduces to

$$\left[\alpha, f\right]_A = \rho\left(\alpha\right) f, \quad \left[\alpha,\beta\right]_A = \left[\alpha,\beta\right], \quad \alpha,\beta \in \Gamma\left(A\right), f \in \mathcal{F}\left(M\right). \tag{4.3}$$

Invoking the generating system of the graded algebra $\tilde{\mathcal{A}}^\bullet_A$ and the natural pairing between $\tilde{\mathcal{A}}^\bullet_A$ and $\mathcal{A}^\bullet_A$ (see Chapter 1), the one-to-one correspondence between the Lie algebroid structure and the homological derivation is captured by

$$\langle \mathrm{d}_A f, \alpha\rangle = \rho\left(\alpha\right) f, \quad \langle \mathrm{d}_A \tilde{\theta}, \alpha\wedge\beta\rangle = \rho\left(\alpha\right)\langle\theta,\beta\rangle - \rho\left(\beta\right)\langle\theta,\alpha\rangle - \langle\theta,[\alpha,\beta]\rangle, \tag{4.4}$$

for arbitrary $f \in \mathcal{F}(M)$, $\tilde{\theta} \in \tilde{\mathcal{A}}^1_A$ and $\alpha,\beta \in \Gamma(A)$.

It is noteworthy that the homological derivation d_A defined in (4.4) is nothing but *de Rham differential* associated with Gerstenhaber structure $[\bullet,\bullet]_A$

$$\langle \mathrm{d}_A\tilde{\omega},\alpha_0\wedge\cdots\wedge\alpha_p\rangle=\sum_{j=0}^{p}(-)^j\rho(\alpha_j)\langle\tilde{\omega},\alpha_0\wedge\cdots\overset{j}{\wedge}\cdots\wedge\alpha_p\rangle + \sum_{0\le i<j\le p}(-)^{i+j}\langle\tilde{\omega},[\alpha_i,\alpha_j]\wedge\alpha_0\wedge\cdots\overset{i}{\wedge}\cdots\overset{j}{\wedge}\cdots\wedge\alpha_p\rangle, \tag{4.5}$$

where by the symbol $\overset{i}{\wedge}$, we meant the omission of the factor α_i in the product $\alpha_0\wedge\cdots\wedge\alpha_p$. With these specifications at hand, the Cartan calculus is ready. Indeed, if we denote by $i_\alpha^{(A)}$ the inner derivation in the algebra $\tilde{\mathcal{A}}^\bullet_A$,

$$i_\alpha^{(A)}f:=0,\quad i_\alpha^{(A)}\tilde{\omega}:=\langle\tilde{\omega},\alpha\rangle,\quad f\in\tilde{\mathcal{A}}^0_A,\tilde{\omega}\in\tilde{\mathcal{A}}^1_A$$

and by $\mathcal{L}_\alpha^{(A)}$ the Lie derivative

$$\mathcal{L}_\alpha^{(A)}f=:[\alpha,f]_A,\quad \langle\mathcal{L}_\alpha^{(A)}\tilde{\omega},\beta\rangle:=\mathcal{L}_\alpha^{(A)}\langle\tilde{\omega},\beta\rangle-\langle\tilde{\omega},[\alpha,\beta]_A\rangle,\quad f\in\tilde{\mathcal{A}}^0_A,\tilde{\omega}\in\tilde{\mathcal{A}}^1_A,\beta\in\Gamma(A)$$

then

$$\mathcal{L}_\alpha^{(A)}=i_\alpha^{(A)}\mathrm{d}_A+\mathrm{d}_A i_\alpha^{(A)}.$$

The Cartan calculus is completed by the graded commutation relations

$$[\mathrm{d}_A,\mathrm{d}_A]=\left[\mathrm{d}_A,\mathcal{L}_\alpha^{(A)}\right]=\left[i_\alpha^{(A)},i_\beta^{(A)}\right]=0,$$

$$\left[\mathcal{L}_\alpha^{(A)},\mathcal{L}_\beta^{(A)}\right]=\mathcal{L}_{[\alpha,\beta]}^{(A)},\quad \left[\mathcal{L}_\alpha^{(A)},i_\beta^{(A)}\right]=i_{[\alpha,\beta]}^{(A)},$$

Remember here that a Lie algebroid $(A\to M,\rho,[\bullet,\bullet])$ is said to be *transitive* if its anchor is surjective, i.e.

$$\operatorname{Im}\rho=TM.$$

Throughout this paper, a key Lie algebroid appears, namely the Lie algebroid of the derivations [43], also known as the Lie algebroid of covariant differential operators [53]. Let $E\to M$ be a vector bundle. There exists the vector bundle $DE\to M$ [73] whose the fiber at $x\in M$, $(DE)_x$, consists in $\mathbb{R}$-linear operators

$$\delta:\Gamma(E)\to E_x, \tag{4.6}$$

which enjoy the existence of tangent vector $\xi\in T_xM$ such that

$$\delta(f\alpha)=(\xi f)\alpha(x)+f(x)\delta\alpha,\quad \alpha\in\Gamma(E),f\in\mathcal{F}(M). \tag{4.7}$$

From the definition in the above, it is clear that property (4.7) guarantees the uniqueness of the tangent vector ξ which is the well-known *symbol* of the operator δ, $\sigma_\delta:=\xi$. The module of sections in the vector bundle $DE\to M$, $\Gamma(DE)$, coincides with the module of the derivations [43] in the vector bundle $E\to M$

$$\Gamma(DE)\equiv\mathcal{D}(E). \tag{4.8}$$

Remember here that an element from the $\mathcal{F}(M)$-module $\mathcal{D}(E)$ is an $\mathbb{R}$-linear map $\Delta : \Gamma(E) \to \Gamma(E)$, covering the derivation X_Δ in the algebra $\mathcal{F}(M)$, $X_\Delta \in \mathfrak{X}^1(M)$, i.e.

$$\Delta(f\alpha) = (X_\Delta f)\alpha + f\Delta\alpha, \quad \alpha \in \Gamma(E), f \in \mathcal{F}(M). \tag{4.9}$$

Identification (4.8) exhibits the Lie algebroid structure on $DE \to M$, $([\bullet,\bullet]_{DE}, \rho_{DE})$, where

$$[\Delta,\Delta']_{DE} := \Delta\Delta' - \Delta'\Delta, \quad \rho_{DE}(\Delta) := X_\Delta, \quad \Delta,\Delta' \in \mathcal{D}(E), \tag{4.10}$$

which is nothing but the well-known *Atiyah algebroid* associated with the vector bundle $E \to M$ [43, 73].

In the last part of this section, we focus on Jacobi algebroids, which appear naturally by 'enriching' Lie algebroids with supplementary data and represent the proper framework for Jacobi-like bundles. This starts with the concept of A-connection on vector bundles [43].

Definition 4.1.2. *Let $([\bullet,\bullet],\rho)$ be a Lie algebroid structure on the vector bundle $A \to M$ and $E \to M$ be another vector bundle. A vector bundle morphism*

$$\nabla : A \to DE \tag{4.11}$$

which enjoys the property

$$\rho_{DE} \circ \nabla = \rho \tag{4.12}$$

is said to be an A-connection on the vector bundle $E \to M$. In addition, the connection ∇ is said to be flat *if its curvature is trivial i.e.*

$$R_\nabla(\alpha,\beta) := [\nabla_\alpha,\nabla_\beta]_{DE} - \nabla_{[\alpha,\beta]} = 0, \quad \alpha,\beta \in \Gamma(A). \tag{4.13}$$

A flat A-connection is known as a representation *of the Lie algebroid A on the vector bundle $E \to M$ [43].*

It is noteworthy that, if in the previous definition, we consider $A = TM$, i.e., the tangent Lie algebroid then a flat connection ∇ on the vector bundle $E \to M$ equips DE with a trivial Lie algebroid structure [43] as

$$DE = TM \oplus (E^* \otimes E).$$

Definition 4.1.3. *Let (A,L) be a pair consisting of a vector bundle $A \to M$ and a line bundle $L \to M$. A triplet $([\bullet,\bullet],\rho,\nabla)$, where $([\bullet,\bullet],\rho)$ is a Lie algebroid structure on the vector bundle $A \to M$ and ∇ is a flat A-connection on the line bundle $L \to M$, is called a Jacobi algebroid structure [73, 32].*

Following Theorem 4.1.1, similar algebraic characterizations can also be done for Jacobi algebroid structures [73, 32].

Theorem 4.1.4. *Let (A,L) be a pair consisting in a vector bundle $A \to M$ and a line bundle $L \to M$. Denoting by $A_L := A \otimes L^*$ the total space of the vector bundle $A \otimes_M L^*$, the following data are equivalent:*

1. *a Jacobi algebroid structure, $([\bullet,\bullet],\rho,\nabla)$, on the pair (A,L);*

2. *a Gerstenhaber-Jacobi algebra structure,* $\left([\bullet,\bullet]_{A,L}\,, X_\bullet^{(A,L)}\right)$*, on the graded module* $\mathcal{L}^\bullet_{A,L} := \Gamma\left(\wedge^\bullet A_L \otimes L\right)[1]$ *over the graded algebra* $\mathcal{A}^\bullet_{A,L} := \Gamma\left(\wedge^\bullet A_L\right)$*;*

3. *a homological degree 1 graded derivation,* $d_{A,L}$ *covering* d_A*, acting on the graded* $\tilde{\mathcal{A}}^\bullet_A$*-module* $\tilde{\mathcal{L}}^\bullet_{A,L} := \Gamma\left(\wedge^\bullet A^* \otimes L\right)$*.*

Remember here that a Gerstenhaber-Jacobi structure consists of the following data:

- a graded module $\mathcal{M}$ over an associative, unital, graded commutative and graded algebra $\mathcal{A}$;
- a degree zero bracket $[\cdot,\cdot]_{\mathcal{M}}$ on $\mathcal{M}$ that organizes $\mathcal{M}$ as an graded $\mathbb{R}$-Lie algebra;
- a degree zero graded $\mathbb{R}$-Lie algebra map $X^{(\mathcal{M})} : \mathcal{M} \to Der(\mathcal{A})$;
- the bracket $[\cdot,\cdot]_{\mathcal{M}}$ is a bi-differential operator in its arguments

$$[m, a\cdot n] = (X_m^{(\mathcal{A})} a)\cdot n + (-)^{|m|\cdot|a|} a\cdot[m,n]\,.$$

In the second statement of Theorem 4.1.4, the argument 1 means that the gradation in the module $\Gamma\left(\wedge^\bullet A_L \otimes L\right)[1]$, $\|\bullet\|$, comes from the natural one in $\Gamma\left(\wedge^\bullet A_L \otimes L\right)$, $|\bullet|$, by shifting it with one unit, $\|\bullet\| = |\bullet| - 1$, i.e.

$$\begin{aligned}\mathcal{A}^\bullet_{A,L} &= \mathcal{A}^0_{A,L} \oplus \mathcal{A}^1_{A,L} \oplus \mathcal{A}^2_{A,L} \oplus \cdots := \mathcal{F}(M) \oplus \Gamma\left(A_L\right) \oplus \Gamma\left(\wedge^2 A_L\right) \oplus \cdots\,,\\ \mathcal{L}^\bullet_{A,L} &= \mathcal{L}^{-1}_{A,L} \oplus \mathcal{L}^0_{A,L} \oplus \mathcal{L}^1_{A,L} \oplus \cdots := \Gamma(L) \oplus \Gamma\left(A_L \otimes L\right) \oplus \Gamma\left(\wedge^2 A_L \otimes L\right) \oplus \cdots\,.\end{aligned}$$

The natural pairing between L and L^* exhibits the isomorphism

$$\mathcal{L}^0_{A,L} := \Gamma\left(A_L \otimes L\right) = \Gamma(A),$$

which allows to display the one-to-one correspondence between the Gerstenhaber-Jacobi algebra and the Jacobi algebroid structures

$$[\alpha,\beta]_{A,L} = [\alpha,\beta]\,, \quad X_\alpha^{(A,L)} f = \rho\left(\alpha\right) f, \quad [\alpha, e]_{A,L} = \nabla_\alpha e, \tag{4.14}$$

with

$$\alpha,\beta \in \Gamma\left(A\right), \quad f \in \mathcal{F}\left(M\right), \quad e \in \Gamma\left(L\right).$$

In order to synthesize the one-to-one relation between Jacobi algebroid structures and homological degree 1 derivations, we use again the natural pairing between L and L^* and interpret the degree k homogeneous elements in the graded algebra $\tilde{\mathcal{A}}^\bullet_A$, $\tilde{\omega} \in \tilde{\mathcal{A}}^k_A$, as the skew-symmetric and multi-linear applications

$$\tilde{\omega} : \Gamma\left(A\right) \times \cdots \times \Gamma\left(A\right) \to \mathcal{F}\left(M\right), \quad \tilde{\omega}\left(\alpha_1,\ldots,\alpha_k\right) := \langle \tilde{\omega}, \alpha_1 \wedge \cdots \wedge \alpha_k\rangle.$$

Invoking the same argument, the degree k homogeneous elements in the graded module $\tilde{\mathcal{L}}^\bullet_{A,L}$, $\omega \in \tilde{\mathcal{L}}^k_{A,L}$, are the skew-symmetric and multi-linear applications

$$\omega : \Gamma\left(A\right) \times \cdots \times \Gamma\left(A\right) \to \Gamma\left(L\right), \quad \omega\left(\alpha_1,\ldots,\alpha_k\right) := \langle \omega, \alpha_1 \wedge \cdots \wedge \alpha_k\rangle.$$

From this perspective, the $\tilde{\mathcal{A}}^{\bullet}_{A}$-module structure of the $\mathbb{R}$-vector space $\tilde{\mathcal{L}}^{\bullet}_{A,L}$ reads

$$\langle \tilde{\omega} \wedge \omega, \alpha_1 \wedge \cdots \wedge \alpha_{k+l} \rangle := \sum_{\sigma \in S(k,l)} (-)^{\sigma} \langle \tilde{\omega}, \alpha_{\sigma(1)} \wedge \cdots \wedge \alpha_{\sigma(k)} \rangle \langle \omega, \alpha_{\sigma(k+1)} \wedge \cdots \wedge \alpha_{\sigma(k+l)} \rangle,$$

where $S(k,l)$ is the subset of (k,l) un-shuffle permutations in $S(k+l)$ i.e. those permutations σ that enjoy of $\sigma(1) < \cdots < \sigma(k)$ and $\sigma(k+1) < \cdots < \sigma(k+l)$.

With these preparations at hand, the one-to-one correspondence between Jacobi algebroid structures and homological degree 1 derivations specified in the previous theorem simply reads

$$(\mathrm{d}_{A,L} e)(\alpha) = \nabla_{\alpha} e, \quad \alpha \in \Gamma(A), e \in \Gamma(L), \tag{4.15}$$

$$\mathrm{d}_{A,L}(\tilde{\omega} \wedge \omega) = (\mathrm{d}_A \tilde{\omega}) \wedge \omega + (-)^k \tilde{\omega} \wedge \mathrm{d}_{A,L} \omega, \quad \tilde{\omega} \in \tilde{\mathcal{A}}^k_A, \omega \in \tilde{\mathcal{L}}^{\bullet}_{A,L} \tag{4.16}$$

where the symbol d_A is related with the Lie algebroid structure on $A \to M$ via (4.5).

It is noteworthy that the homological degree 1 derivation $\mathrm{d}_{A,L}$ allows the Cartan calculus on the $\tilde{\mathcal{A}}^{\bullet}_{A}$-module $\tilde{\mathcal{L}}^{\bullet}_{A,L}$. Indeed, for any $\alpha \in \Gamma(A)$, we define the degree -1 derivation $\iota^{(A,L)}_{\alpha}$ covering $i^{(A)}_{\alpha}$, $\iota^{(A,L)}_{\alpha} : \tilde{\mathcal{L}}^{\bullet}_{A,L} \to \tilde{\mathcal{L}}^{\bullet}_{A,L}$, via

$$\iota^{(A,L)}_{\alpha} e := 0, \quad \iota^{(A,L)}_{\alpha} (\tilde{\omega} \otimes e) := \langle \tilde{\omega}, \alpha \rangle e, \quad \tilde{\omega} \in \Gamma(A^*), e \in \Gamma(L).$$

Also, the degree 0 derivation $\mathcal{L}^{(A,L)}_{\alpha}$ covering $\mathcal{L}^{(A)}_{\alpha}$, $\mathcal{L}^{(A,L)}_{\alpha} : \tilde{\mathcal{L}}^{\bullet}_{A,L} \to \tilde{\mathcal{L}}^{\bullet}_{A,L}$, is available

$$\mathcal{L}^{(A,L)}_{\alpha} e := \nabla_{\alpha} e, \quad \mathcal{L}^{(A,L)}_{\alpha} (\tilde{\omega} \otimes e) := \left(\mathcal{L}^{(A)}_{\alpha} \tilde{\omega}\right) \otimes e + \tilde{\omega} \otimes \nabla_{\alpha} e, \quad \tilde{\omega} \in \Gamma(A^*), e \in \Gamma(L).$$

The previous derivations enjoy

$$\mathcal{L}^{(A,L)}_{\alpha} = \iota^{(A,L)}_{\alpha} \mathrm{d}_{A,L} + \mathrm{d}_{A,L} \iota^{(A,L)}_{\alpha}$$

together with the commutation relations

$$[\mathrm{d}_{A,L}, \mathrm{d}_{A,L}] = \left[\mathrm{d}_{A,L}, \mathcal{L}^{(A,L)}_{\alpha}\right] = \left[\iota^{(A,L)}_{\alpha}, \iota^{(A,L)}_{\beta}\right] = 0,$$

$$\left[\mathcal{L}^{(A,L)}_{\alpha}, \mathcal{L}^{(A,L)}_{\beta}\right] = \mathcal{L}^{(A,L)}_{[\alpha,\beta]_A}, \quad \left[\mathcal{L}^{(A,L)}_{\alpha}, \iota^{(A,L)}_{\beta}\right] = \iota^{(A,L)}_{[\alpha,\beta]_A},$$

specific to Cartan calculus. Previously, by $[\bullet, \bullet]$ we understood the graded commutator between various $\mathbb{R}$-linear maps.

Remark 4.1.5. *The definition of Jacobi algebroid structure* $([\bullet, \bullet], \rho, \nabla)$ *when the line bundle is trivial*

$$L = \mathbb{R}_M := \mathbb{R} \times M \tag{4.17}$$

reduces to that of a Lie algebroid structure $([\bullet, \bullet], \rho)$ *with* a 1-cocycle *[39]. Indeed, the trivial line bundle (4.17) displays [43]*

$$\Gamma(\mathbb{R}_M) = \mathcal{F}(M), \quad \mathcal{D}(\mathbb{R}_M) = \mathfrak{X}^1(M) \oplus \mathcal{F}(M), \tag{4.18}$$

which further exhibits a 1-form $\omega_\nabla \in \Gamma(A^*)$ that enjoys

$$\nabla_\alpha = \rho(\alpha) + \langle \omega_\nabla, \alpha \rangle, \quad \alpha \in \Gamma(A). \tag{4.19}$$

Moreover, the flatness of A-connection (4.19) is equivalent

$$d_A \omega_\nabla = 0. \tag{4.20}$$

In the same context (4.17), the graded algebra $\mathcal{A}^\bullet_{A,L}$ and the graded module $\mathcal{L}^\bullet_{A,L}$ become

$$\mathcal{A}^\bullet_{A,L} = \mathcal{A}^\bullet_A := \Gamma(\wedge^\bullet A), \quad \mathcal{L}^\bullet_{A,L} = \mathcal{A}^\bullet_A[1], \tag{4.21}$$

while the Gerstenhaber-Jacobi algebra structure $\left([\bullet,\bullet]_{A,L}, X_\bullet^{(A,L)}\right)$ reduces to $\left([\bullet,\bullet]_A^\nabla, X_\bullet^\nabla\right)$, where

$$[\alpha, f]_A^\nabla = \nabla_\alpha f, \quad [\alpha,\beta]_A^\nabla = [\alpha,\beta], \quad X_\alpha^\nabla f = \rho(\alpha) f, \quad \alpha,\beta \in \Gamma(A), f \in \mathcal{F}(M). \tag{4.22}$$

Finally, the $\tilde{\mathcal{A}}^\bullet_A := \Gamma(\wedge^\bullet A^*)$-module $\tilde{\mathcal{L}}^\bullet_{A,L} := \Gamma(\wedge^\bullet A^* \otimes L)$ reads

$$\tilde{\mathcal{L}}^\bullet_{A,L} = \tilde{\mathcal{A}}^\bullet_A, \tag{4.23}$$

while the homological derivation $d_{A,L}$ (covering d_A) becomes d_A^∇

$$d_A^\nabla \omega = d_A \omega + \omega_\nabla \wedge \omega. \tag{4.24}$$

By direct computation it can be proved that homological derivation (4.24) coincides with de Rham differential associated with graded Lie algebra structure (4.21) i.e.

$$\begin{aligned} \langle d_A^\nabla \omega, \alpha_0 \wedge \cdots \wedge \alpha_p \rangle = & \sum_{j=0}^{p} (-)^j \left[\alpha_j, \langle \omega, \alpha_0 \wedge \cdots \overset{j}{\wedge} \cdots \alpha_p \rangle\right]_A^\nabla \\ & + \sum_{0 \le i < j \le p} (-)^{i+j} \langle \omega, [\alpha_i,\alpha_j]_A^\nabla \wedge \alpha_0 \wedge \cdots \overset{i}{\wedge} \cdots \overset{j}{\wedge} \cdots \wedge \alpha_p \rangle. \end{aligned} \tag{4.25}$$

Let $L \to M$ be a line bundle. The pair (DL, L) can be naturally endowed with a Jacobi algebroid structure associated with the standard Lie algebroid one (4.10) supplemented with the tautological representation of $DL \to M$ on the line bundle $L \to M$,

$$\nabla : DL \to DL, \quad \nabla_\Box e := \Box e, \quad \Box \in \mathcal{D}(L), e \in \Gamma(L). \tag{4.26}$$

According to Theorem 4.1.4, if we adopt the notations

$$[\bullet,\bullet] := [\bullet,\bullet]_{DL}, \quad \sigma := \rho_{DL}, \tag{4.27}$$

the Jacobi algebroid structure $([\bullet,\bullet], \sigma, \nabla)$ is equivalent to the Gerstenhaber-Jacobi algebra one $\left([\bullet,\bullet] := [\bullet,\bullet]_{DL,L}, X_\bullet := X_\bullet^{(DL,L)}\right)$, on the graded $\mathcal{A}^\bullet_{DL,L} := \Gamma(\wedge^\bullet DL_L)$-module $\mathcal{L}^\bullet_{DL,L} := \Gamma(\wedge^\bullet DL_L \otimes L)[1]$.

At this stage, it is useful to express the previous abstract Gerstenhaber-Jacobi algebra in a more convenient form that allows computations. This assumes both the realization of the algebra $\mathcal{A}^{\bullet}_{DL,L}$ and of the $\mathbb{R}$-vector space $\mathcal{L}^{\bullet}_{DL,L}$. In view of this, we make use of the isomorphism [59] of $\mathcal{F}(M)$-modules

$$\Gamma\left(\left(J^1L\right)^* \otimes L\right) \simeq \Gamma\left(DL\right) := \mathcal{D}\left(L\right). \tag{4.28}$$

At the level of vector bundles, (4.28) is equivalent to the isomorphism

$$\left(J^1L\right)^* \otimes L \simeq DL,$$

which, in the light of the natural pairing between L and L^*, further displays

$$DL_L := DL \otimes L^* \simeq J_1L := \left(J^1L\right)^*. \tag{4.29}$$

It has been shown [59] that if $\{E_l \to M : l = \overline{1,p}\}$ and $F \to M$ are some vector bundles over the same manifold, then there exists the isomorphism of $\mathcal{F}(M)$-modules

$$\Gamma(J_kE_1 \otimes \cdots J_kE_p \otimes F) \simeq \mathcal{D}\mathit{iff}_k(E_1, \cdots, E_p; F), \tag{4.30}$$

where J_kE is the dual of the k-th order jet bundle, $J_kE = (J^kE)^*$ and $\mathcal{D}\mathit{iff}_k(E_1, \cdots, E_p; F)$ is the set of k-th order differential operators in each entry

$$D : \Gamma(E_1) \times \cdots \times \Gamma(E_p) \to \Gamma(F)$$

Putting together results (4.29) and (4.30), we display the realization that we are looking for

$$\Gamma\left(\wedge^{\bullet} J_1L\right) \simeq \mathcal{D}\mathit{iff}^{\bullet}_1\left(L; \mathbb{R}_M\right), \tag{4.31}$$

that further exhibits

$$\mathcal{A}^{\bullet}_{DL,L} \simeq \mathcal{D}\mathit{iff}^{\bullet}_1\left(L; \mathbb{R}_M\right) \Leftrightarrow \mathcal{A}^{k}_{DL,L} \simeq \mathcal{D}\mathit{iff}^{k}_1\left(L; \mathbb{R}_M\right), \quad k \in \mathbb{N}. \tag{4.32}$$

Previously, $\mathcal{D}\mathit{iff}^{k}_1\left(L; \mathbb{R}_M\right)$ consists in the $\mathbb{R}$ multi-linear applications

$$\tilde{\Box} : \Gamma\left(L\right) \times \cdots \times \Gamma\left(L\right) \to \mathcal{F}\left(M\right), \quad \tilde{\Box}\left(e_1, \ldots, e_k\right) \in \mathcal{F}\left(M\right), \tag{4.33}$$

that are skew-symmetric and first-order differential operators in each argument. The product in the graded algebra $\mathcal{D}\mathit{iff}^{\bullet}_1\left(L; \mathbb{R}_M\right)$ is nothing but the exterior one

$$\left(\tilde{\Box} \wedge \tilde{\triangle}\right)\left(e_1, \ldots, e_{k+l}\right) = \sum_{\sigma \in S(k,l)} \left(-\right)^{\sigma} \tilde{\Box}\left(e_{\sigma(1)}, \ldots, e_{\sigma(k)}\right) \tilde{\triangle}\left(e_{\sigma(k+1)}, \ldots, e_{\sigma(k+l)}\right),$$

for arbitrary homogeneous elements $\tilde{\Box} \in \mathcal{D}\mathit{iff}^{k}_1\left(L; \mathbb{R}_M\right)$ and $\tilde{\triangle} \in \mathcal{D}\mathit{iff}^{l}_1\left(L; \mathbb{R}_M\right)$. Previously, we denoted by $S\left(k, l\right)$ the subset of $\left(k, l\right)$ un-shuffle permutations in $S\left(k + l\right)$.

Concerning the realization of the $\mathbb{R}$-vector space $\mathcal{L}^{\bullet}_{DL,L}$, by considering the tensor product of the isomorphism (4.32) with the module $\Gamma(L)$, one immediately gets

$$\mathcal{L}^{\bullet}_{DL,L} \simeq \mathcal{D}^{\bullet}L\left[1\right] := \mathcal{D}\mathit{iff}^{\bullet}_1\left(L; L\right)\left[1\right] \Leftrightarrow \mathcal{L}^{k}_{DL,L} \simeq \mathcal{D}^{k+1}L, \quad k \geq -1 \tag{4.34}$$

where $\mathcal{D}^{k+1}L$ consists in the $\mathbb{R}$-multi-linear applications

$$\Box : \Gamma(L) \times \cdots \times \Gamma(L) \to \Gamma(L), \quad \Box\,(e_1, \ldots, e_{k+1}) \in \Gamma(L), \tag{4.35}$$

that are skew-symmetric and first-order differential operators in each argument.

At this stage, the left action of the graded algebra $\mathcal{D}iff_1^\bullet(L;\mathbb{R}_M)$ on the $\mathbb{R}$-vector space $\mathcal{D}^\bullet L$ is nothing but the wedge product

$$\left(\tilde{\triangle} \wedge \Box\right)(e_1, \ldots, e_{k+l+1}) = \sum_{\sigma \in S(k,l+1)} (-)^\sigma \, \tilde{\triangle}\left(e_{\sigma(1)}, \ldots, e_{\sigma(k)}\right) \Box \left(e_{\sigma(k+1)}, \ldots, e_{\sigma(k+l+1)}\right),$$

where $\tilde{\triangle} \in \mathcal{D}iff_1^k(L;\mathbb{R}_M)$ and $\Box \in \mathcal{D}^{l+1}L$ are arbitrary homogeneous elements. Moreover, the pair $(\mathcal{D}iff_1^\bullet(L;\mathbb{R}_M), \mathcal{D}^\bullet L)$ has a natural Gerstenhaber-Jacobi algebra structure $([\![\bullet,\bullet]\!], \mathbb{X}_\bullet)$ as follows. The graded Lie algebra structure on $\mathcal{D}^\bullet L$, $[\![\bullet,\bullet]\!]$, can be written in terms of the Gerstenhaber inner multiplication [32]

$$\Box \circ \triangle\,(e_1, \ldots, e_{k+l+1}) := \sum_{\sigma \in S(l+1,k)} (-)^\sigma \, \Box\left(\triangle\left(e_{\sigma(1)}, \ldots, e_{\sigma(l+1)}\right), e_{\sigma(l+2)}, \ldots, e_{\sigma(k+l+1)}\right)$$

as

$$[\![\Box, \triangle]\!] := (-)^{kl}\, \Box \circ \triangle - \triangle \circ \Box, \quad \Box \in \mathcal{D}^{k+1}L, \triangle \in \mathcal{D}^{l+1}L. \tag{4.36}$$

In order to introduce the derivative representation of the module $\mathcal{D}^\bullet L$ on the graded algebra $\mathcal{D}iff_1^\bullet(L;\mathbb{R}_M)$, $\mathbb{X}_\bullet$, we define the *symbol map*

$$\sigma_\Box(f)(e_1, \ldots, e_k)\, e := \Box\,(fe, e_1, \ldots, e_k) - f\Box\,(e, e_1, \ldots, e_k), \tag{4.37}$$

where

$$\Box \in \mathcal{D}^{k+1}L, \quad f \in \mathcal{F}(M), \quad e, e_1, \ldots, e_k \in \Gamma(L).$$

It is noteworthy that, in the light of the pairing between L and L^*, the symbol $\sigma_\Box(f)(e_1, \ldots, e_k)$ is just a smooth function on the manifold M

$$\sigma_\Box(f)(e_1, \ldots, e_k) \in \Gamma(L^* \otimes L) \simeq \mathcal{F}(M),$$

and moreover $\sigma_\Box(f) \in \mathcal{D}iff_1^k(L;\mathbb{R}_M)$. With these specifications at hand, the derivative representation reads

$$\begin{aligned} \mathbb{X}_\Box\left(\tilde{\triangle}\right)(e_1, \ldots, e_{k+l}) &= (-)^{k(l-1)} \sum_{\sigma \in S(l,k)} (-)^\sigma \, \sigma_\Box\left(\tilde{\triangle}\left(e_{\sigma(1)}, \ldots, e_{\sigma(l)}\right)\right)\left(e_{\sigma(l+1)}, \ldots, e_{\sigma(l+k)}\right) \\ &- \sum_{\sigma \in S(k+1,l-1)} (-)^\sigma \, \tilde{\triangle}\left(\Box\left(e_{\sigma(1)}, \ldots, e_{\sigma(k+1)}\right), e_{\sigma(k+2)}, \ldots, e_{\sigma(k+l)}\right). \end{aligned} \tag{4.38}$$

Finally, in the same context of the pair (DL, L), Theorem 4.1.4 offers the equivalence between the Lie algebroid structure $([\bullet,\bullet], \sigma, \nabla)$ and the homological degree 1 graded derivation, $\mathrm{d}_L := \mathrm{d}_{DL,L}$ covering d_{DL}, acting on the graded $\tilde{\mathcal{A}}^\bullet_{DL} := \Gamma(\wedge^\bullet (DL)^*)$-module $\Omega^\bullet_L := \tilde{\mathcal{L}}^\bullet_{DL,L} := \Gamma(\wedge^\bullet (DL)^* \otimes L)$. In literature [67], de Rham complex $(\Omega^\bullet_L, \mathrm{d}_L)$ is known as *der-complex* associated with the line bundle $L \to M$ and, meanwhile, the homogeneous elements of the module $\Omega^\bullet_L$ are called *L-valued Atiyah forms*. Regarding the homological derivation

d_L, by means of the general results (4.15)–(4.16), it acts on the homogeneous elements of the der-complex via

$$\langle \mathrm{d}_L e, \square\rangle := \square e, \quad e \in \Gamma(L), \square \in \mathcal{D}(L), \tag{4.39}$$

$$\mathrm{d}_L(\tilde{\omega}\wedge\omega) = \mathrm{d}_{DL}\tilde{\omega}\wedge\omega + (-)^k \tilde{\omega}\wedge \mathrm{d}_L\omega, \quad \tilde{\omega}\in\tilde{\mathcal{A}}^k_{DL}, \omega\in\Omega^\bullet_L. \tag{4.40}$$

Remark 4.1.6. *The homological derivation enjoys two strong properties: i) it* agrees *with the first-order prolongation and ii) it is* acyclic*.*
i) Indeed, the isomorphism (4.28) with the concrete expression

$$\mathcal{I}:\Gamma\left(\left(J^1L\right)^*\otimes L\right)\to\mathcal{D}(L), \quad (\mathcal{I}\varphi)\, e := \varphi\left(j^1 e\right),$$

where $j^1 : \Gamma(L) \to \Gamma(J^1L)$ is the first-order prolongation [69], furnishes the L-pairing between DL and J^1L expressed by the bi-linear non-degenerate map

$$\langle\bullet,\bullet\rangle : \mathcal{D}(L)\times\Gamma(J^1L)\to\Gamma(L), \quad \langle\square, j^1e\rangle := \square e, \tag{4.41}$$

which is well-defined as the $\mathcal{F}(M)$-module $\Gamma(J^1L)$ is generated [69] by $\operatorname{Im} j^1$. In the light of (4.41) *it results that*

$$\langle d_L e, \square\rangle = \langle\square, j^1 e\rangle.$$

ii) The acyclicity [67] of the homological derivation d_L is done by the existence of a contracting homotopy for $\mathrm{id}_{\Omega^\bullet_L}$ with respect to d_L. Indeed, by direct computation it can be shown that the Lie derivative associated with the first-order differential operator

$$\mathbb{1}\in\mathcal{D}(L), \quad \mathbb{1}e := e, \quad e\in\Gamma(L),$$

$\mathcal{L}^{(DL,L)}_{\mathbb{1}}$ reduces to $id_{\Omega^\bullet_L}$ which means that

$$\iota^{(DL,L)}_{\mathbb{1}} d_L + d_L\iota^{(DL,L)}_{\mathbb{1}} = id_{\Omega^\bullet_L}. \tag{4.42}$$

Remark 4.1.7. *When we consider the trivial line bundle (4.17) then, using the outputs* (4.1)*, it results that the pair $(D\mathbb{R}_M, \mathbb{R}_M)$ is nothing but $(TM\oplus\mathbb{R}_M, \mathbb{R}_M)$. In this case, the Jacobi algebroid structure* (4.26)*–*(4.27) *reduces to*

$$[(X,f),(Y,g)] = ([X,Y], Xg - Yf), \quad (X,f),(Y,g)\in\mathfrak{X}^1(M)\oplus\mathcal{F}(M), \tag{4.43}$$

$$\sigma((X,f)) = X, \quad (X,f)\in\mathfrak{X}^1(M)\oplus\mathcal{F}(M), \tag{4.44}$$

$$\nabla_{(X,f)}h = Xh + fh, \quad (X,f)\in\mathfrak{X}^1(M)\oplus\mathcal{F}(M), h\in\mathcal{F}(M). \tag{4.45}$$

Comparing expression (4.45) *with the* 1*-cocycle given by Remark 4.1.5 it results that*

$$\omega_\nabla = (0,1)\in\Gamma((TM\oplus\mathbb{R})^*) = \Omega^1(M)\oplus\mathcal{F}(M). \tag{4.46}$$

According to point 2 in Theorem 4.1.4, the previous Jacobi algebroid structure $([\bullet,\bullet],\sigma,\nabla)$ over the pair $(D\mathbb{R}_M,\mathbb{R}_M)$ is equivalent to a Gerstenhaber-Jacobi structure over the $\mathcal{D}iff^\bullet_1(\mathbb{R}_M;\mathbb{R}_M)$-module $\mathcal{D}iff^\bullet_1(\mathbb{R}_M;\mathbb{R}_M)$ (see realizations (4.32) *and* (4.34)*). Moreover, the graded algebra of multi-derivations $\mathcal{D}iff^\bullet_1(\mathbb{R}_M;\mathbb{R}_M)$ admits the realization*

$$\mathfrak{X}^k(M)\oplus\mathfrak{X}^{k-1}(M)\simeq\mathcal{D}iff^k_1(\mathbb{R}_M;\mathbb{R}_M), \quad (P,Q)\longleftrightarrow P - Q\wedge\mathrm{id}, \tag{4.47}$$

where

$$(P - Q\wedge \mathrm{id})(f_1,\dots,f_k) := P(f_1,\dots,f_k) - \sum_{j=1}^{k}(-)^{k-j}Q\left(f_1,\dots,\widehat{f_j},\dots,f_k\right)f_j. \quad (4.48)$$

The Gerstenhaber-Jacobi structure [62] $\left([\![\bullet,\bullet]\!]^{(0,1)},\mathbb{X}_\bullet^{(0,1)}\right)$ *consists in*

$$\begin{aligned}[\![P - Q\wedge \mathrm{id}, R - S\wedge \mathrm{id}]\!]^{(0,1)} &= [P,R] - p(-)^r P\wedge S + rQ\wedge R \\ &\quad - \left([P,S] + (-)^r[Q,R] - (p-r)Q\wedge S\right)\wedge \mathrm{id},\end{aligned} \quad (4.49)$$

and

$$\mathbb{X}^{(0,1)}_{P-Q\wedge \mathrm{id}}(R - S\wedge \mathrm{id}) = [\![P - Q\wedge \mathrm{id}, R - S\wedge \mathrm{id}]\!]^{(0,1)} + Q\wedge R - (Q\wedge S)\wedge \mathrm{id}\,. \quad (4.50)$$

Finally, invoking point 3 in Theorem 4.1.4, it results that the Jacobi algebroid structure (4.43)–(4.45) *is equivalent to* der-complex $\Omega^{\bullet}_{\mathbb{R}_M}$ *endowed with the homological degree* 1 *derivation* $\boldsymbol{d}^{(0,1)}$ *covering de Rham differential* $\boldsymbol{d}$ *(that differential associated with the Lie algebroid structure* (4.43)–(4.44) *over* $TM\oplus\mathbb{R}_M$*). The homogeneous elements from* $\Omega^{\bullet}_{\mathbb{R}_M}$ *are the multi-linear and skew-symmetric applications*

$$\tilde{\omega} : \left(\mathfrak{X}^1(M)\oplus\mathcal{F}(M)\right)\times\cdots\times\left(\mathfrak{X}^1(M)\oplus\mathcal{F}(M)\right)\to\mathcal{F}(M)\,.$$

But the $\mathcal{F}(M)$*-module of* k*-multi-linear applications in the above is isomorphic to* $\Omega^k(M)\oplus\Omega^{k-1}(M)$

$$\Omega^k(M)\oplus\Omega^{k-1}(M)\simeq\Omega^{k}_{\mathbb{R}_M},\quad \left(\overset{(k)}{\omega},\overset{(k-1)}{\omega}\right)\longleftrightarrow\overset{(k)}{\omega} + \mathrm{id}\wedge\overset{(k-1)}{\omega}, \quad (4.51)$$

where

$$\begin{aligned}\left(\overset{(k)}{\omega} + \mathrm{id}\wedge\overset{(k-1)}{\omega}\right)\left((X_1,f_1),\dots,(X_k,f_k)\right) &:= \langle\overset{(k)}{\omega}, X_1\wedge\cdots\wedge X_k\rangle + \\ + \sum_{j=1}^{k}(-)^{j-1}\langle\overset{(k-1)}{\omega}, X_1\wedge\cdots\overset{j}{\wedge}\cdots\wedge X_k\rangle f_j.\end{aligned} \quad (4.52)$$

Furthermore, the homological derivation of degree 1 *in der-complex,* $\boldsymbol{d}^{(0,1)}$*, becomes*

$$\boldsymbol{d}^{(0,1)}\left(\overset{(k)}{\omega} + \mathrm{id}\wedge\overset{(k-1)}{\omega}\right) = d\overset{(k)}{\omega} + \mathrm{id}\wedge\left(-d\overset{(k-1)}{\omega} + \overset{(k)}{\omega}\right), \quad (4.53)$$

where d *is the standard de Rham differential associated with the manifold* M*.*

Remark 4.1.8. *Comparing the structures* (4.49) *(derived via the Gerstenhaber inner multiplication through* (4.36)*) and* (3.8) *(inferred from* (3.4) *via* (3.8)*), we conclude that they coincide, as we expected.*

Also, the relation (4.53) *explains the provenience of definition* (3.10)*.*

4.2 Jacobi bundles and Jacobi pairs

Let $L \to M$ be a line bundle. By its very definition [56], a *Jacobi structure* on the considered line bundle is an $\mathbb{R}$-Lie algebra structure on $\Gamma(L)$, $\{\bullet, \bullet\}$, which is a derivation in both of its arguments,

$$\{\bullet, e\} \in \mathcal{D}(L), \quad e \in \Gamma(L).$$

It is noteworthy that such a structure is nothing but a *local Lie algebra* [42] on the line bundle $L \to M$. With these specifications, let's fix the terminology. By definition, a *Jacobi bundle* is a line bundle endowed with a Jacobi structure over it, $(L \to M, \{\bullet, \bullet\})$ while a *Jacobi manifold* is a manifold equipped with a Jacobi bundle over it.

The previous definition places the Jacobi structures into the framework of Jacobi algebroid structure $([\bullet, \bullet], \sigma, \nabla)$ over the pair (DL, L). Indeed, if we use the notation $J := \{\bullet, \bullet\}$

$$J(e_1, e_2) = \{e_1, e_2\}, \quad e_1, e_2 \in \Gamma(L), \tag{4.54}$$

then $J \in \mathcal{D}^2 L$. Moreover, using the Gerstenhaber inner multiplication, direct computations yields

$$(J \circ J)(e_1, e_2, e_3) = \{\{e_1, e_2\}, e_3\} + \{\{e_2, e_3\}, e_1\} + \{\{e_3, e_1\}, e_2\} := -\mathrm{Jac}\{e_1, e_2, e_3\}, \tag{4.55}$$

which further exhibits

$$[\![J, J]\!](e_1, e_2, e_3) = 2\mathrm{Jac}\{e_1, e_2, e_3\}.$$

The last equality shows that a Jacobi structure on a line bundle $L \to M$ consists of the bi-differential operator $J \in \mathcal{D}^2 L$ subject to the Maurer-Cartan equation

$$[\![J, J]\!] = 0. \tag{4.56}$$

So, from now on, a Jacobi bundle is addressed in terms of the pair $(L \to M, J)$ with J a bi-differential operator, $J \in \mathcal{D}^2 L$, satisfying (4.56).

Remark 4.2.1. *When the line bundle is trivial* (4.17)*, according with Remark 4.1.7, a bi-differential operator $J \in \mathcal{D}^2 \mathbb{R}_M$ is expressed in terms of a pair (Π, E) as $J = \Pi - E \wedge \mathrm{id}$ (see* (4.48)*). With this expression at hand, the Gerstenhaber-Jacobi bracket* (4.49) *implies that equation* (4.56) *is equivalent to*

$$[\Pi, \Pi] + 2\Pi \wedge E = 0, \quad [\Pi, E] = 0. \tag{4.57}$$

With this expression of the bi-differential operator J, the $\mathbb{R}$-Lie algebra structure over $\mathcal{F}(M)$ (see the first formula in (4.18)*), $\{\bullet, \bullet\}$, reduces to the well-known Jacobi bracket*

$$\{f, g\} = i_\Pi (df \wedge dg) + i_E (f dg - g df), \quad f, g \in \mathcal{F}(M).$$

A smooth manifold M equipped with a pair $(\Pi, E) \in \mathfrak{X}^2(M) \times \mathfrak{X}^1(M)$ that verifies equations (4.57) *(the well-known* Jacobi pair*) is said to be a* Jacobi manifold *[33].*

In the light of the previous remark, we conclude that *Jacobi pairs are in one-to-one correspondence with trivial Jacobi bundles.*

Let $(L \to M, J)$ be a Jacobi bundle. By means of the isomorphisms (4.34), the bi-differential operator $J \in \mathcal{D}^2 L$ exhibits (via the fact that the module $\Gamma(J^1 L)$ is generated by $\operatorname{Im} j^1$) the element $\hat{J} \in \Gamma\left(\wedge^2 J_1 L \otimes L\right)$

$$\langle j^1 e_1 \wedge j^1 e_2, \hat{J}\rangle := J\left(e_1, e_2\right), \quad e_1, e_2 \in \Gamma(L) \tag{4.58}$$

which further displays the morphism of $\mathcal{F}(M)$-modules

$$\hat{J}^\sharp : \Gamma\left(J^1 L\right) \to \mathcal{D}(L), \quad \hat{J}^\sharp(j^1 e_1) e_2 := J\left(e_1, e_2\right), \quad e_1, e_2 \in \Gamma(L). \tag{4.59}$$

This morphism allows the introduction of a smooth distribution on the base manifold

$$\mathcal{K}_J := \operatorname{Im}\left(\sigma \circ \hat{J}^\sharp\right), \tag{4.60}$$

known as *the characteristic distribution* of the considered Jacobi bundle. By definition, the considered Jacobi bundle is said to be *transitive* if the characteristic distribution coincides with the tangent space of the base manifold

$$\mathcal{K}_J = TM, \tag{4.61}$$

or, equivalently, the vector bundle map $\sigma \circ \hat{J}^\sharp : J^1 L \to TM$ is surjective.

It is noteworthy that the transitiveness in the present context is strongly related with the transitivity [73] of a specific Jacobi algebroid associated with the considered Jacobi bundle. This is due to the fact that *any* Jacobi structure $J := \{\bullet, \bullet\}$ on a given line bundle $L \to M$ *is equivalent to* a Jacobi algebroid structure $([\bullet, \bullet]_J, \rho_J, \nabla^J)$ on the pair $(J^1 L, L)$ (see Proposition 3.4 in [23] or Propositions 2.2 and 2.3 in [73]). The Jacobi structure J organizes $J^1 L$ as a Lie algebroid with respect to

$$\left[j^1 e_1, j^1 e_2\right]_J := j^1 J\left(e_1, e_2\right), \quad \rho_J(j^1 e) := X_e,$$

where X_e is the symbol (the well-known *Hamiltonian vector field*) corresponding to the *Hamiltonian derivation* $\triangle_e$, $\triangle_e \in \mathcal{D}(L)$, associated with the section $e \in \Gamma(L)$

$$\triangle_e(e_1) := J\left(e, e_1\right), \quad e_1 \in \Gamma(L), \tag{4.62}$$

i.e.,

$$X_e := \rho\left(\triangle_e\right). \tag{4.63}$$

Moreover, in the light of axiom (4.56), which is equivalent to

$$[\triangle_{e_1}, \triangle_{e_2}] = \triangle_{\{e_1, e_2\}},$$

it results that the Lie algebroid $(J^1 L, [\bullet, \bullet]_J, \rho_J)$ enjoys the natural representation on the line bundle L,

$$\nabla^J : J^1 L \to DL, \quad \nabla^J_{j^1 e} := \triangle_e.$$

With these specifications at hand, it is clear that the characteristic distributions $\mathcal{K}_J$ and $\operatorname{Im} \rho_J$ of the Jacobi line bundle $(L \to M, J)$ and respectively of the Jacobi algebroid $\left(J^1 L, L, [\bullet, \bullet]_J, \rho_J, \nabla^J\right)$ coincide. As an immediate consequence, it results that the transitivity of starting Jacobi line bundle is equivalent to that of the associated Jacobi algebroid. This allows the integrability result given below.

Theorem 4.2.2. *The characteristic distribution of a Jacobi structure $J \in \mathcal{D}^2 L$ on the line bundle $L \to M$ is completely integrable [71, 72] with the characteristic leaves equipped with transitive Jacobi structures induced by J.*

Transitive Jacobi bundles. A deep analysis [73] of transitive Jacobi bundles over even-dimensional base manifolds has shown that these are in one-to-one correspondence with locally conformal symplectic structures (lcs for short) over the same base manifolds.

Definition 4.2.3. *By its very definition [73], an lcs structure on a given line bundle $L \to M$ is a pair (∇, ω) consisting in a representation ∇ of the tangent Lie algebroid $(TM \to M, [\bullet,\bullet], \mathrm{id})$ on a line bundle and a non-degenerate L-valued 2-form $\omega \in \Omega^2(M; L)$ which is closed with respect to the homological degree 1 derivation d_∇ associated with the Jacobi algebroid structure $([\bullet,\bullet], \mathrm{id}, \nabla)$ on the pair (TM, L) (see the third statement in Theorem 4.1.4)*

$$\mathrm{d}_\nabla \omega = 0.$$

In the light of the previous definition, it is easy to associate a Jacobi structure to a given lcs one. Indeed, the non-degeneracy of the 2-form ω displays the vector bundle isomorphism

$$\omega^\flat : TM \to T^*M \otimes L, \quad \omega^\flat X := -i_X \omega, \tag{4.64}$$

with i_X the standard right inner multiplication [51] by vectors. Denoting by $\omega^\sharp$ its inverse, we introduce the Hamiltonian vector fields by

$$X_e := \omega^\sharp \left(\mathrm{d}_\nabla e\right), \quad e \in \Gamma(L), \tag{4.65}$$

which exhibits the first-order differential operator

$$X_\bullet : \Gamma(L) \to TM, \quad e \mapsto X_e \tag{4.66}$$

Indeed, putting together definition (4.65) with the Leibniz rule verified by the homological degree 1 derivation d_∇ covering d

$$\mathrm{d}_\nabla(fe) = \mathrm{d}f \otimes e + f \mathrm{d}_\nabla e, \quad f \in \mathcal{F}(M),$$

it results that

$$X_{fe} - f X_e = \omega^\sharp(\mathrm{d}f \otimes e), \quad f \in \mathcal{F}(M), e \in \Gamma(L).$$

At this stage, we define

$$J : \Gamma(L) \times \Gamma(L) \to \Gamma(L), \quad J(e_1, e_2) := \langle \omega, X_{e_1} \wedge X_{e_2} \rangle, \quad e_1, e_2 \in \Gamma(L), \tag{4.67}$$

which is manifestly skew-symmetric and, due to (4.66), is a first-order differential operator in each entry, i.e., $J \in \mathcal{D}^2 L$.

The previous definition together with the closedness of the 2-form ω prove that the bi-differential operator J verifies (4.56), i.e., it is a Jacobi structure on the line bundle.

In order to synthesize the converse relation, we introduce the *bi-symbol* of a given bi-differential operator $J \in \mathcal{D}^2 L$. In view of this, we start from the Spencer short exact sequence

$$\Omega^1(M; L) \overset{\gamma}{\rightarrowtail} \Gamma(J^1 L) \overset{\pi_{1,0}}{\twoheadrightarrow} \Gamma(L), \quad \gamma\left(\mathrm{d}f \otimes e\right) := j^1(fe) - f j^1(e), \quad \pi_{1,0}(f j^1(e)) := fe$$

(which particulary splits as short exact sequence of $\mathbb{R}$-vector spaces by the first-order prolongation $j^1 : \Gamma(L) \to \Gamma(J^1 L)$) and consistently define the bi-symbol of J, $\tilde{J} \in \Gamma\left(\wedge^2 (T^*M \otimes L)^* \otimes L\right)$, via

$$\langle \eta \wedge \theta, \tilde{J}\rangle := \langle \gamma(\eta) \wedge \gamma(\theta), \hat{J}\rangle, \quad \eta, \theta \in \Omega^1(M; L). \tag{4.68}$$

Using the natural pairing $L^* \otimes L = \mathbb{R}_M$, the bi-symbol $\tilde{J}$ displays the vector bundle morphism

$$\tilde{J}^\sharp : T^*M \otimes L \to TM, \quad \langle \theta_2, \tilde{J}^\sharp(\theta_1 \otimes e_1)\rangle e_2 := \langle (\theta_1 \otimes e_1) \wedge (\theta_2 \otimes e_2), \tilde{J}\rangle \tag{4.69}$$

which enjoys

$$\tilde{J}^\sharp = \sigma \circ \hat{J}^\sharp \circ \gamma. \tag{4.70}$$

At this stage, it becomes transparent that the bi-symbol of the bi-differential operator (4.67) coincides with the inverse of the vector bundle isomorphism (4.64), $\tilde{J}^\sharp = \omega^\sharp$. This result is a key clue for the converse implication proof. Indeed, a simple dimension counting shows that if $(L \to M, J)$ is a transitive Jacobi bundle over an even-dimensional base manifold then, its bi-symbol, $\tilde{J}^\sharp$, is non-degenerate, i.e., it is a vector bundle isomorphism. This allows the construction of the non-degenerate 2-form $\omega \in \Omega^2(M; L)$ via

$$\langle \omega, \tilde{J}\theta \wedge \tilde{J}\eta\rangle := \langle \theta \wedge \eta, \tilde{J}\rangle.$$

Invoking again the transitivity, one constructs the correspondence

$$\nabla_{X_e} := \triangle_e,$$

which is nothing but a representation of the tangent Lie algebroid $(TM \to M, [\bullet, \bullet], \mathrm{id})$ on the line bundle. By direct computation [73], it results that (∇, ω) is a locally conformal symplectic structure on the line bundle $L \to M$.

Remark 4.2.4. *It is noteworthy that, in the context of trivial line bundles, the resulting Jacobi pairs are in one-to-one correspondence with locally conformal symplectic ones [76].*

Remark 4.2.5. *From the flow of the previous argumentation, we can conclude that if $(L \to M, J)$ is a Jacobi bundle over an even-dimensional base manifold, then it is transitive, if and only if the bi-symbol $\tilde{J}$ is non-degenerate.*

Concerning transitive Jacobi bundles over odd-dimensional base manifolds, these were shown to be in a one-to one-correspondence with contact structures over the same base manifolds.

Definition 4.2.6. *A contact structure over a smooth manifold (necessarily odd-dimensional) M is a hyperplane distribution $\mathcal{H} \subset TM$ which is* maximally non-integrable *i.e. its curvature*

$$\omega_{\mathcal{H}} : \mathcal{H} \times \mathcal{H} \to L := TM/\mathcal{H}, \quad \omega_{\mathcal{H}}(X, Y) := [X, Y] \mod \mathcal{H} \tag{4.71}$$

is non-degenerate, i.e., the linear map

$$\omega^\flat_{\mathcal{H}} : \Gamma(\mathcal{H}) \to \Gamma(\mathcal{H}^* \otimes L), \quad \langle \omega^\flat_{\mathcal{H}} X, Y\rangle := -\omega_{\mathcal{H}}(X, Y)$$

is invertible.

It is worth noticing that a contact structure can be defined, in a dual view, in terms of a non-trivial canonical projection

$$\theta \in \Omega^1(M;L), \quad \langle\theta, X\rangle := X \mod \mathcal{H}, \tag{4.72}$$

which is nothing but the well-known *contact* 1-*form*. From this perspective, it results that (4.71) is nothing but the curvature of the contact 1-form

$$\omega_{\mathcal{H}} = -(\mathrm{d}_\nabla\theta)|_{\operatorname{Ker}\theta} \Longleftrightarrow \langle\omega_{\mathcal{H}}, X\wedge Y\rangle = \langle\theta, [X,Y]\rangle, \quad X, Y \in \Gamma(\mathcal{H}), \tag{4.73}$$

with respect to an *arbitrary* TM-connection on the line bundle $L \to M$, $\nabla : TM \to DL$.

Motivated by a similar analysis that will be done for Jacobi bundles with background, we will sketch here the well-known correspondence between contact structures and transitive Jacobi bundles over odd-dimensional manifolds [23, 73].

Let $\mathcal{H}$ be a contact distribution on M. In this context, there exists the decomposition of $\mathbb{R}$-vector spaces

$$\mathfrak{X}^1(M) = \mathfrak{X}_{\mathcal{H}} \oplus \Gamma(\mathcal{H}), \tag{4.74}$$

where $\mathfrak{X}_{\mathcal{H}}$ is the $\mathbb{R}$-Lie subalgebra

$$X \in \mathfrak{X}_{\mathcal{H}} \Leftrightarrow [X, \Gamma(\mathcal{H})] \subset \Gamma(\mathcal{H}), \tag{4.75}$$

of the $\mathbb{R}$-Lie algebra $\mathfrak{X}^1(M)$ whose elements are the well-known Reeb vector fields [23](or contact vector fields [73]). The previous decomposition comes from the splitting $\omega^\sharp_{\mathcal{H}}$: $\Gamma(\mathcal{H}^*\otimes L) \to \Gamma(\mathcal{H})$ of the short exact sequence of vector spaces

$$\mathfrak{X}_{\mathcal{H}} \overset{\subseteq}{\rightarrowtail} \mathfrak{X}^1(M) \overset{\varphi}{\twoheadrightarrow} \Gamma(\mathcal{H}^*\otimes L),$$

where

$$\mathfrak{X}^1(M) \ni X \to \varphi_X \in \Gamma(\mathcal{H}^*\otimes L), \quad \varphi_X(Y) := -\langle\theta, [X,Y]\rangle. \tag{4.76}$$

It is clear that φ is a first-order differential operator

$$\varphi_{fX} = \mathrm{d}f|_{\mathcal{H}} \otimes \langle\theta, X\rangle + f\varphi_X, \tag{4.77}$$

which reduces to the inverse of $\omega^\sharp_{\mathcal{H}}$, $\omega^\flat_{\mathcal{H}}$, when restricted to $\Gamma(\mathcal{H})$,

$$\varphi_X = \omega^\flat_{\mathcal{H}} X, \quad X \in \Gamma(\mathcal{H}).$$

Now, putting together definition (4.72) and decomposition (4.74), it results that the contact 1-form is invertible when restricted to $\mathfrak{X}_{\mathcal{H}}$. This further exhibits the $\mathbb{R}$-linear operator $X : \Gamma(L) \to \mathfrak{X}^1(M)$ that associates to each section in the line bundle $e \in \Gamma(L)$ the unique vector field $X_e \in \mathfrak{X}_{\mathcal{H}} \subset \mathfrak{X}^1(M)$ enjoying

$$\langle\theta, X_e\rangle = e. \tag{4.78}$$

Moreover, by particularizing (4.77) to Reeb vector fields (4.78) it results that $\varphi_{fX_e} = \mathrm{d}f|_{\mathcal{H}}\otimes e$, which combined with the splitting $\omega_{\mathcal{H}}$ eventually show that X is also a first-order differential operator satisfying

$$X_{fe} = fX_e - \omega^\sharp_{\mathcal{H}}(\mathrm{d}f|_{\mathcal{H}} \otimes e). \tag{4.79}$$

The vector fields X_e associated with smooth sections in the line bundle $e \in \Gamma(L)$ are nothing but the Hamiltonian vector fields. By means of these ingredients, the bi-differential operator that we are looking for reads

$$J_{\mathcal{H}}(e_1, e_2) := \langle \theta, [X_{e_1}, X_{e_2}] \rangle, \quad e_1, e_2 \in \Gamma(L).$$

This definition combined with result (4.79) prove that Hamiltonian vector fields associated with induced Jacobi structure $J_{\mathcal{H}}$ are generated (via multiplication with the elements from commutative algebra $\mathcal{F}(M)$) by Reeb vector fields, i.e., the induced Jacobi structure is transitive.

Conversely, let $(L \to M, J)$ be a transitive Jacobi bundle over the odd-dimensional smooth manifold M, $\dim M = 2m+1$. Due to the fact that $\hat{J}^\sharp$ comes from a skew-symmetric linear map, it results that $\mathrm{Im}\,\hat{J}^\sharp$ is even-dimensional. On the other hand, combining the relation $\mathrm{Im}(\sigma \circ \hat{J}) = \sigma(\mathrm{Im}\,\hat{J})$ with the assumed transitivity further gives

$$2m + 2 = \dim(J_1 L \otimes L) \geq \dim(\mathrm{Im}\,\hat{J}^\sharp) = 2m + 1 + \dim(\mathrm{Im}\,\hat{J}^\sharp \cap \mathrm{Ker}\,\sigma),$$

which exhibits

$$\dim(\mathrm{Im}\,\hat{J}^\sharp \cap \mathrm{Ker}\,\sigma) = 1 \Rightarrow \dim(\mathrm{Im}\,\hat{J}) = 2m + 2$$

i.e. the surjectivity of $\hat{J}^\sharp$ and, moreover, the bijectivity of $\hat{J}^\sharp$. At the same time, due to the fact that map (4.69) comes from the skew-symmetric one (4.68) implies that $\mathrm{Im}\,\tilde{J}^\sharp$ is even-dimensional. In addition, a dimension counting based on (4.70), $\dim \mathrm{Im}\,\tilde{J}^\sharp = \dim \mathrm{Im}(\sigma \circ \hat{J}^\sharp) - \dim(\mathrm{Ker}(\sigma \circ \hat{J}^\sharp) \cap \mathrm{Im}\,\gamma)$, supplemented with the surjectivity of the map $\sigma \circ \hat{J}$ leads to: i) $\mathrm{Ker}(\sigma \circ \hat{J}) \subseteq \mathrm{Im}\,\gamma$ and ii) $\mathrm{Im}\,\tilde{J}^\sharp$ is a hyperplane distribution on M. Let's denote this distribution by

$$\mathcal{H}_J := \mathrm{Im}\,\tilde{J}^\sharp = (\sigma \circ \hat{J})(\mathrm{Im}\,\gamma). \tag{4.80}$$

In the standard manner [51], this can be endowed with an L-valued 2-form, $\omega_{\tilde{J}} \in \Gamma\left(\wedge^2 \mathcal{H}^* \otimes L\right)$ via

$$\langle \omega_{\tilde{J}}, \tilde{J}^\sharp \alpha \wedge \tilde{J}^\sharp \beta \rangle := \tilde{J}(\alpha, \beta), \quad \alpha, \beta \in \Omega^1(M; L),$$

which is non-degenerate. It remains to show that $\omega_{\tilde{J}}$ is nothing but the curvature of the hyperplane distribution

$$\omega_{\tilde{J}} = \omega_{\mathcal{H}_J},$$

where $\omega_{\mathcal{H}_J}$ is associated with the hyperplane distribution (4.80) via (4.73). To do so, we show that the $\mathbb{R}$-vector space decomposition (4.74) takes place. This is done by constructing an appropriate short exact sequence of $\mathbb{R}$-vector spaces

$$\Gamma(\mathcal{H}_J) \overset{i_{\mathcal{H}_J}}{\rightarrowtail} \mathfrak{X}^1(M) \overset{\hat{X}}{\twoheadrightarrow} \underline{\mathfrak{ham}}(L \to M, J), \tag{4.81}$$

which, admits a split. Previously, we denoted by $\underline{\mathfrak{ham}}(L \to M, J)$ the set of Hamiltonian vector fields generated by the considered Jacobi structure

$$\underline{\mathfrak{ham}}(L \to M, J) := \mathrm{Im}\left(\sigma \circ \hat{J}^\sharp \circ j^1\right).$$

Also, by $i_{\mathcal{H}_J}$ we mean the inclusion vector bundle morphism, $i_{\mathcal{H}_J} : \Gamma(\mathcal{H}_J) \to \mathfrak{X}^1(M)$.

At this stage, we introduce the isomorphism of $\mathbb{R}$-vector spaces

$$X : \Gamma(L) \to \underline{\mathfrak{ham}}(L \to M, J), \quad X := \sigma \circ \hat{J}^\sharp \circ j^1,$$

that enjoys of

$$X_{fe} = \tilde{J}^\sharp(\mathrm{d}f \otimes e) + f X_e, \quad f \in \mathcal{F}(M), \quad e \in \Gamma(L).$$

This isomorphism, supplemented with Spencer short exact sequence of $\mathcal{F}(M)$-modules

$$\Omega^1(M; L) \overset{\gamma}{\rightarrowtail} \Gamma(J^1 L) \overset{\pi_{1,0}}{\twoheadrightarrow} \Gamma(L), \tag{4.82}$$

and

$$i_{\mathcal{H}_J} \circ \tilde{J}^\sharp = \sigma \circ \hat{J}^\sharp \circ \gamma, \tag{4.83}$$

allow to close (via the universality of quotient space) the $\mathbb{R}$-vector spaces diagram

$$\begin{array}{ccccc} \Omega^1(M; L) & \xrightarrow{\gamma} & \Gamma(J^1 L) & \xrightarrow{\pi_{1,0}} & \Gamma(L) \\ \downarrow{\scriptstyle \tilde{J}^\sharp} & & \downarrow{\scriptstyle \sigma \circ \hat{J}^\sharp} & & \downarrow{\scriptstyle X} \\ \Gamma(\mathcal{H}_J) & \xrightarrow{i_{\mathcal{H}_J}} & \mathfrak{X}^1(M) & \xrightarrow{\hat{X}} & \underline{\mathfrak{ham}}(L \to M, J) \end{array} \tag{4.84}$$

Invoking the fact that the first-order prolongation, j^1, splits the top line in (4.84), $j^1 \circ \pi_{1,0} = \mathrm{id}_{\Gamma(L)}$, it results that the down line (which is also a short exact sequence of $\mathbb{R}$-vector spaces) in the same diagram admits a split too. This is nothing but the inclusion of Hamiltonian vector fields into the algebra of smooth vector fields

$$\hat{X} \circ i_{\mathfrak{ham}} = \mathrm{id}_{\mathfrak{ham}},$$

that finally exhibits

$$\mathfrak{X}^1(M) = \Gamma(\mathcal{H}_J) \oplus \underline{\mathfrak{ham}}(L \to M, J).$$

Remark 4.2.7. *If the line bundle is trivial then the resulting Jacobi pairs are in one-to-one correspondence with coorientable contact structures [64, 76].*

Remark 4.2.8. *From the flow of the previous argumentation, we can conclude that if $(L \to M, J)$ is a Jacobi bundle over an odd-dimensional base manifold, then it is transitive, if and only if the linear map $\hat{J}$ is non-degenerate.*

The previous results concerning transitive Jacobi bundles can be synthesized as follows.

Theorem 4.2.9. *If a Jacobi structure $J \in \mathcal{D}^2 L$ on the line bundle $L \to M$ is transitive then M is either a locally conformal symplectic structure (if the base manifold is even-dimensional) or a contact one (if the base manifold is odd-dimensional) on the same line bundle.*

Jacobi bundle maps. The category of Jacobi bundles is completed by morphisms of Jacobi bundles, i.e., Jacobi maps. Let $(L_i \to M_i, \{\bullet, \bullet\}_i)$, $i = 1, 2$ be two Jacobi bundles. A regular vector bundle morphism (i.e. fiber-wise isomorphism) $\varphi : L_1 \to L_2$ covering $\underline{\varphi} \in \mathcal{C}^\infty(M_1, M_2)$ is said to be a Jacobi bundle map [56] iff

$$\varphi^*\{\lambda, \mu\}_2 = \{\varphi^*\lambda, \varphi^*\mu\}_1, \quad \lambda, \mu \in \Gamma(L_2). \tag{4.85}$$

Previously, by φ^* we denoted the pull-back associated with the given regular vector bundle morphism

$$\varphi^* : \Gamma(L_2) \to \Gamma(L_1), \quad (\varphi^*\mu)(x) := (\varphi_x)^{-1} \mu\left(\underline{\varphi}(x)\right), \quad x \in M_1.$$

4.3 Twisted Jacobi bundles and twisted Jacobi pairs

In this part we relax the Maurer-Cartan equation (4.56) by twisting it via a closed L-valued Atiyah 3-form $\Psi \in \Omega_L^3 := \Gamma\left(\wedge^3 (DL)^* \otimes L\right)$,

$$\mathrm{d}_L \Psi = 0, \tag{4.86}$$

i.e,

$$[\![J, J]\!] = 2\left(\wedge^3 \hat{J}^\sharp\right)^* \Psi, \tag{4.87}$$

where by $\wedge^3 \hat{J}^\sharp$ we meant the linear (with respect to the commutative algebra $\mathcal{F}(M)$) extension of map (4.59),

$$\wedge^3 \hat{J}^\sharp : \Gamma\left(\wedge^3 J^1 L\right) \to \Gamma\left(\wedge^3 DL\right), \quad \wedge^3 \hat{J}^\sharp(j^1 e_1, j^1 e_2, j^1 e_3) := \hat{J}^\sharp(j^1 e_1) \wedge \hat{J}^\sharp(j^1 e_2) \wedge \hat{J}^\sharp(j^1 e_3).$$

In addition, in the light of isomorphisms (4.34), the object in the left-hand side of (4.87) is an element from $\Gamma\left(\wedge^3 J_1 L \otimes L\right)$.

According to Remark 4.1.6, der-complex $(\Omega_L^\bullet, \mathrm{d}_L)$ is acyclic. This means that there exists the L-valued Atiyah 2-form $\Omega \in \Omega_L^2$ such that $\Psi = \mathrm{d}_L \Omega$. With these prolegomena at hand, we can introduce the concept of the twisted Jacobi bundle in its full generality.

Definition 4.3.1. *A twisted Jacobi bundle is a triple $(L \to M, J, \Omega)$ consisting in a line bundle $L \to M$, a first-order bi-differential operator $J \in \mathcal{D}^2 L$ and an L-valued Atiyah 2-form $\Omega \in \Omega_L^2$ that verify the consistency condition*

$$[\![J, J]\!] = 2\left(\wedge^3 \hat{J}^\sharp\right)^* d_L \Omega. \tag{4.88}$$

Extending the terminology adopted for Jacobi manifolds, we say that a given smooth manifold M is a *twisted Jacobi* one if it is the base manifold for a twisted Jacobi bundle.

In order to show that twisted Jacobi bundles encompass twisted Jacobi manifolds [62, 64], we are going to briefly address the trivial line bundle situation (4.17). Using Remark 4.1.7, the bi-differential operator J becomes

$$J = \Pi - E \wedge \mathrm{id}, \tag{4.89}$$

while its 'hat' associated morphism reads

$$\hat{J}^\sharp : \Omega^1(M) \oplus \mathcal{F}(M) \to \mathfrak{X}^1(M) \oplus \mathcal{F}(M), \quad \hat{J}^\sharp(\theta + f \wedge \mathrm{id}) = \Pi^\sharp \theta + fE - (i_E \theta) \wedge \mathrm{id}\,. \tag{4.90}$$

Direct computation based on (4.90) exhibits the expression of $\wedge^3 \hat{J}^\sharp$

$$\wedge^3 \hat{J}^\sharp(\psi + \omega \wedge \mathrm{id}) = \wedge^3 \Pi^\sharp \psi + \wedge^2 \Pi^\sharp \omega \wedge E - \left(\wedge^2 \Pi^\sharp i_E \psi + \Pi^\sharp i_E \omega \wedge E\right) \wedge \mathrm{id}, \tag{4.91}$$

with $\wedge^\bullet \Pi^\sharp$ the linear extensions of

$$\Pi^\sharp : \Omega^1(M) \to \mathfrak{X}^1(M), \quad \Pi^\sharp \alpha := -j_\alpha \Pi.$$

Invoking the definition of the homological degree 1 derivation (4.53), we can always choose the Atiyah 2-form Ω as

$$\Omega = \omega \in \Omega^2(M)$$

such that

$$\mathbf{d}^{(0,1)}\omega = \mathrm{d}\omega + \mathrm{id}\wedge\omega. \tag{4.92}$$

Putting together the information in (4.89)–(4.92) into the twisted Maurer-Cartan equation (4.88) one displays the equations

$$\tfrac{1}{2}[\Pi,\Pi] + E\wedge\Pi = \wedge^3\Pi^\sharp \mathrm{d}\omega + \wedge^2\Pi^\sharp\omega\wedge E, \quad [E,\Pi] = -\left(\wedge^2\Pi^\sharp i_E \mathrm{d}\omega + \Pi^\sharp i_E\omega\wedge E\right), \tag{4.93}$$

which exhibit a twisted Jacobi structure [62, 64] on the base manifold M.

In the light of the previous analysis, we conclude that twisted Jacobi pairs (see Definition 3.1.1) are in one-to-one correspondence with twisted Jacobi bundles over trivial line bundles.

Like in the standard Jacobi bundles (see Section 4.2), we introduce the characteristic distribution associated with a twisted Jacobi bundle $(L\to M, J, \Omega)$ via

$$\mathcal{K}_J := \mathrm{Im}\,\sigma\circ\hat{J}^\sharp \tag{4.94}$$

and say that the considered twisted Jacobi bundle is *transitive* if

$$\mathcal{K}_J = TM. \tag{4.95}$$

In the sequel we are focusing on the characterization of transitive twisted Jacobi bundles. For doing this, we initially introduce the twisted versions of locally conformal symplectic structures and contact ones.

Definition 4.3.2. *A twisted locally conformal symplectic structure on a given line bundle $L\to M$ is a pair $((\nabla,\omega),\hat{\omega})$ consisting in a representation ∇ of the tangent Lie algebroid $(TM\to M, [\bullet,\bullet], \mathrm{id})$ on a line bundle and two L-valued 2-forms $\omega,\hat{\omega}\in\Omega^2(M;L)$, among which ω is non-degenerate and verifies the consistency condition*

$$d_\nabla(\omega-\hat{\omega}) = 0. \tag{4.96}$$

Previously, by d_∇ we denoted the homological degree 1 derivation associated with the Jacobi algebroid structure $([\bullet,\bullet], \mathrm{id}, \nabla)$ on the pair (TM, L) (see the third statement in Theorem 4.1.4).

It is worth noticing that in the trivial line bundle context (4.17), a twisted locally conformal symplectic structure reduces to a twisted locally conformal symplectic pair [19, 20]. The manifold endowed with such a pair being nothing but a twisted locally conformal symplectic one [62, 64]. Indeed, when the line bundle is trivial (4.17), according to Remark 4.1.5 the flat connection ∇ is given by the closed 1-form $\alpha\in\Omega^1(M)$, $\alpha := \omega_\nabla$ via (4.19). Clearly, here closedness refers to de Rham differential, which coincides with homological degree 1 derivation associated with the Lie algebroid $(TM\to M, [\bullet,\bullet], \mathrm{id})$ (see third statement in Theorem 4.1.1). Moreover, in the same context, the module $\Omega^\bullet(M;L)$ over the graded commutative algebra $\Omega^\bullet(M)$ reduces to $\Omega^\bullet(M)$ over the same graded commutative algebra, which further implies that the L-valued 2-forms in Definition 4.3.2 are ordinary ones, $\omega,\hat{\omega}\in\Omega^2(M)$. Within this framework, the homological degree 1 derivation (4.25), which will be denoted by d_α, reduces to

$$\mathrm{d}_\alpha\theta = \mathrm{d}\theta + \alpha\wedge\theta, \quad \alpha\in\Omega^\bullet(M). \tag{4.97}$$

The previous derivation is nothing but the well-known Morse-Novikov derivation [6]. With all of these at hand, the considered twisted locally conformal symplectic structure reduces to the pair $((\omega, \alpha), \hat{\omega})$ with ω a non-degenerate 2-form that verifies the consistency condition

$$\mathrm{d}(\omega - \hat{\omega}) + \alpha \wedge (\omega - \hat{\omega}) = 0,$$

i.e., a twisted locally conformal symplectic pair [16, 62, 64].

Definition 4.3.3. *A twisted contact structure on a manifold M consists of a hyperplane distribution $\mathcal{H} \subset TM$ and an L-valued 2-form $\psi \in \Omega^2(M; L)$ such that*

$$\Omega := \omega_{\mathcal{H}} + \psi|_{\mathcal{H}} \in \Gamma\left(\wedge^2 \mathcal{H}^* \otimes L\right) \tag{4.98}$$

is non-degenerate. In (4.98)*, $\omega_{\mathcal{H}}$ is the curvature* (4.71) *of the considered hyperplane distribution $\mathcal{H}$.*

It is noteworthy that the structure previously introduced has been recently addressed [75] as *almost contact structure*.

Due to the skew-symmetry of Ω, its non-degeneracy necessarily implies that $\mathcal{H}$ is even-dimensional, which is equivalent with the fact that the base manifold M is odd-dimensional, $\dim M = 2m + 1$.

We briefly show that Definition 4.3.3 encompasses the twisted cooriented [28] contact manifolds [64]. For doing this, we assume that the hyperplane distribution comes from the contact 1-form $\theta \in \Omega^1(M)$ via

$$\mathcal{H} = \operatorname{Ker} \theta, \tag{4.99}$$

which further leads to the isomorphism

$$L = \mathbb{R}_M$$

exhibited (via the universality property of the quotient vector bundle) by the surjective vector bundle morphism

$$\theta : TM \to \mathbb{R}_M, \quad X_p \to \langle \theta_p, X_p \rangle.$$

Based on this isomorphism, the L-valued 2-forms from Definition 4.3.3 can be interpreted as elements in $\Gamma\left(\wedge^2 \mathcal{H}^*\right)$. Particulary, by means of definition (4.71) it results that

$$\omega_{\mathcal{H}} = -\, \mathrm{d}\theta|_{\mathcal{H}}\,. \tag{4.100}$$

At this stage we use a splitting of the short exact sequence of vector bundles [59]

$$\mathcal{H} \overset{i_{\mathcal{H}}}{\rightarrowtail} TM \overset{\theta}{\twoheadrightarrow} \mathbb{R}_M,$$

i.e., an injective vector bundle map $\epsilon : \mathbb{R}_M \to TM$ which enjoys the property

$$\theta \circ \epsilon = \mathrm{id}_{\mathbb{R}_M}\,.$$

The injective morphism is equivalent with a nowhere vanishing vector field $E \in (TM)^\times$, which satisfies

$$\langle \theta, E \rangle = 1.$$

Using the previous vector field, we introduce the vector bundle map

$$p_{\mathcal{H}} : TM \to \mathcal{H}, \quad p_{\mathcal{H}}(X) := X - \langle \theta, X \rangle E$$

which is surjective and enjoys the property

$$p_{\mathcal{H}} \circ i_{\mathcal{H}} = \mathrm{id}_{\mathcal{H}} \, .$$

Finally, we construct the 2-forms $\omega := p_{\mathcal{H}}^* \omega_{\mathcal{H}}$, $\tilde{\omega} := p_{\mathcal{H}}^* \psi$ and $\tilde{\Omega} := p_{\mathcal{H}}^* \Omega = \omega + \tilde{\omega}$. By construction, it is clear that

$$\tilde{\Omega}^\flat = p_{\mathcal{H}}^* \Omega^\flat p_{\mathcal{H}},$$

which, in the light of the non-degeneracy of Ω, further leads to

$$\mathrm{Im}\, \tilde{\Omega}^\flat = \mathrm{Im}\, p_{\mathcal{H}}^* = (\mathrm{Ker}\, p_{\mathcal{H}})^\circ = \langle E \rangle^\circ .$$

The last result shows that $T^*M = \mathrm{Im}\, \tilde{\Omega}^\flat \oplus \langle \theta \rangle$, which exhibits the volume form $\theta \wedge \tilde{\Omega}^m$. This result supplemented with (4.100) display also the volume form

$$\mu := \theta \wedge (\mathrm{d}\theta - \tilde{\omega})^m \, , \tag{4.101}$$

i.e., the considered distribution is a twisted cooriented [28] contact one [64].

At this stage, we remember that in the context of trivial line bundle it has been shown [62, 64]that twisted Jacobi bundles are in one-to-one correspondence either with twisted locally conformal symplectic structures on the same line bundles (i.e. twisted locally conformal symplectic pairs) when the base manifold is even-dimensional or with twisted cooriented contact structures on the same base manifolds when the latter are odd-dimensional. This result can be generalized to arbitrary line bundles as follows.

Theorem 4.3.4. *Let $(L \to M, J, \Omega)$ be a transitive twisted Jacobi bundle. Then the following alternative holds.*

- *If the base manifold is even-dimensional then the considered Jacobi bundle is equivalent to a twisted locally conformal symplectic structure on the same line bundle.*
- *If the base manifold is odd-dimensional then the considered Jacobi bundle is equivalent to a twisted contact structure displaying the same line bundle.*

Proof. This will be done in the general context of Jacobi bundles with background (see below). □

Moreover, in the same context (trivial line bundle), the characteristic distributions associated with twisted Jacobi bundles have been proved to be completely integrable [62, 64].

In [64] (see Theorem 3.2) the proof uses the involutivity of the characteristic distribution and the Jacobi bracket associated with the considered twisted Jacobi pair.

In [62], the proof has been done in two steps. Initially, the omni-Lie algebroid associated with the trivial line bundle (4.17)

$$\mathbb{D}\mathbb{R}_M = (TM \times \mathbb{R}) \oplus (T^*M \times \mathbb{R})$$

is organized as a $\mathbf{d}^{(0,1)}\Omega$-twisted Courant-Jacobi algebroid. Then, it is shown that

$$\mathrm{graph}\hat{J}^{\sharp} := \left(\hat{J}^{\sharp}\left(\alpha + f\wedge \mathrm{id}\right), \alpha + f \wedge \mathrm{id}\right),$$

with $\hat{J}^{\sharp}$ given in (4.90), is a Dirac-Jacobi subbundle in the previous Courant-Jacobi algebroid. Due to the fact that its characteristic distribution coincides with $\mathrm{Im}\,\sigma\circ\hat{J}^{\sharp}$ it results the integrability [22] of the characteristic distribution associated with the starting twisted Jacobi bundle. Based on the recent results concerning Dirac-Jacobi bundles [79], the previous algorithm can also be implemented for a generic twisted Jacobi bundle $(L\to M, J, \Omega)$, where the omni-Lie algebroid becomes

$$\mathbb{D}L = DL \oplus J^1 L$$

while its $\mathrm{d}_L\Omega$-twisted Courant-Jacobi algebroid structure consists in the Dorfman-like bracket

$$[\![(\Box_1,\mu_1),(\Box_2,\mu_2)]\!]_{\mathrm{d}_L\Omega} := \left([\Box_1,\Box_2], \mathcal{L}^{(DL,L)}_{\Box_1}\mu_2 - \iota^{(DL,L)}_{\Box_2}\mathrm{d}_L\mu_1 + \langle \mathrm{d}_L\Omega, \Box_1\wedge\Box_2\wedge\bullet\rangle\right),$$

the non-degenerate metric

$$\langle\langle(\Box_1,\mu_1),(\Box_2,\mu_2)\rangle\rangle := \langle\Box_1,\mu_2\rangle + \langle\Box_2,\mu_1\rangle \tag{4.102}$$

and the vector bundle morphism

$$\mathrm{p}_D : DL\oplus J^1L \to DL, \quad \mathrm{p}_D(\Box,\mu) := \Box.$$

In definition (4.102), we denoted by $\langle\bullet,\bullet\rangle$ the L-pairing between DL and J^1L given in (4.41). In the light of this result, the following theorem holds.

Theorem 4.3.5. *Let $(L\to M, J, \Omega)$ be a twisted Jacobi bundle. Then its characteristic distribution is completely Stefan-Sussmann integrable with the characteristic leaves transitive twisted Jacobi manifolds.*

4.4 Jacobi bundles with background and Jacobi manifolds with background

In this section, we will do a complete characterization, as best as we can, of a Jacobi-like bundle, which comes from (4.87) when we give up on the closedness condition (4.86).

Definition 4.4.1. *A* Jacobi bundle with a background 3-form, *shortly a* Jacobi bundle with background, *is a triple $(L\to M, J, \Psi)$ consisting of a line bundle $L\to M$, a first-order bi-differential operator $J\in\mathcal{D}^2L$, and an L-valued Atiyah 3-form $\Psi\in\Omega^3_L$ that verify the consistency condition*

$$[\![J,J]\!] = 2\left(\wedge^3\hat{J}^{\sharp}\right)^*\Psi. \tag{4.103}$$

Extending the terminology adopted for (twisted) Jacobi manifolds, we say that a given smooth manifold M is a *Jacobi manifold with background* if it is the base manifold for a Jacobi bundle with background.

To prove that Jacobi bundles with background encompass Jacobi manifolds with backgrounds [16] (see Chapter 3), we analyze the outputs of the previous definition in the context of the trivial line bundle (4.17). In view of this, we use Remark 4.1.7 and exhibit for the bi-differential operator J expression (4.89). Furthermore, the 'hat' morphisms $\hat{J}^\sharp$ and $\wedge^3\hat{J}^\sharp$ are given by (4.90) and (4.91) respectively. In addition, by means of the isomorphism (4.51), the Atiyah 3-form Φ reads

$$\Psi = \phi + \omega \wedge \mathrm{id}, \quad \phi \in \Omega^3(M), \quad \omega \in \Omega^2(M). \tag{4.104}$$

By inserting the previous information in the consistency equation (4.103), one derives the equations

$$\tfrac{1}{2}\,[\Pi, \Pi] + E \wedge \Pi = \wedge^3\Pi^\sharp\phi + \wedge^2\Pi^\sharp\omega \wedge E, \quad [E, \Pi] = -\left(\wedge^2\Pi^\sharp i_E\phi + \Pi^\sharp i_E\omega \wedge E\right), \tag{4.105}$$

which is nothing but a Jacobi pair (Π, E) with the background (ϕ, ω) on the base manifold M [16]. Remember that Jacobi pairs with background have been scrutinized in the previous chapter.

The outputs displayed in the trivial line bundle setting allow us to conclude that *Jacobi pairs with background* are in one-to-one correspondence with *Jacobi bundles with background over trivial line bundles*.

As in the Jacobi/ twisted Jacobi bundles (see Sections 4.2 and 4.3), we define the characteristic distribution corresponding to a Jacobi bundle with background $(L \to M, J, \Psi)$ via (4.94). Moreover, we say that the considered Jacobi bundle with background is *transitive* if relation (4.95) takes place.

Let $(L \to M, J, \Psi)$ be a Jacobi bundle with background. Its characteristic distribution is strongly related with the Hamiltonian vector fields generated by the first-order bi-differential operator $J \in \mathcal{D}^2L$. These vector fields are symbols associated with Hamiltonian derivations

$$\triangle_e \in \mathcal{D}^1L, \quad \triangle_e := \hat{J}^\sharp(j^1e), \quad e \in \Gamma(L), \tag{4.106}$$

i.e.,

$$X_e := \sigma(\triangle_e) \tag{4.107}$$

Indeed, comparing the definition of characteristic distribution (4.94) with that of the Hamiltonian vector fields (4.108), we get

$$\mathcal{K}_J = \bigcup\{X_e : e \in \Gamma(L)\}. \tag{4.108}$$

Proposition 4.4.2. *The characteristic distribution $\mathcal{K}_J$ corresponding to a Jacobi bundle with background $(L \to M, J, \Psi)$ is involutive.*

Proof. Let $\{\bullet, \bullet\}$ be the $\mathbb{R}$-linear and skew-symmetric bracket associated with the first-order bi-differential operator J,

$$\{e_1, e_2\} := J(e_1, e_2) = \triangle_{e_1}e_2. \tag{4.109}$$

In the light of definition (4.109), result (4.55) leads to

$$\frac{1}{2}[\![J, J]\!](e_1, e_2, e) = [\triangle_{e_1}, \triangle_{e_2}]e - \triangle_{\{e_1,e_2\}}e, \quad e, e_1, e_2 \in \Gamma(L) \tag{4.110}$$

On the other hand, by means of the consistency equation (4.103), the previous relation gives

$$[\triangle_{e_1}, \triangle_{e_2}]e - \triangle_{\{e_1,e_2\}}e = \langle \Psi, \triangle_{e_1} \wedge \triangle_{e_2} \wedge \triangle_e \rangle,$$

or, equivalently

$$[\triangle_{e_1}, \triangle_{e_2}]e - \triangle_{\{e_1,e_2\}}e = \langle \iota^{(DL,L)}_{\triangle_{e_2}} \iota^{(DL,L)}_{\triangle_{e_1}} \Psi, \triangle_e \rangle. \tag{4.111}$$

The second term in the right hand side of the previous equality can be related to the image of $\hat{J}^\sharp$. Indeed, if we use the notation $\Phi := \iota^{(DL,L)}_{\triangle_{e_2}} \iota^{(DL,L)}_{\triangle_{e_1}} \Psi$, then $\Phi \in \Omega^1_L = \Gamma(J^1 L)$. In addition, due to the skew-symmetry of $\hat{J}$ (see definition (4.58)), combined with the first point in Remark 4.1.6, it results that

$$\langle \Phi, \hat{J}^\sharp(j^1 e) \rangle = \langle j^1 e \wedge \Phi, \hat{J} \rangle = -\langle \Phi \wedge j^1 e, \hat{J} \rangle = -\hat{J}^\sharp(\Phi) e \tag{4.112}$$

Putting together results (4.111) and (4.112), we infer

$$[\triangle_{e_1}, \triangle_{e_2}] = \triangle_{\{e_1,e_2\}} + \hat{J}^\sharp \left(\iota^{(DL,L)}_{\triangle_{e_2}} \iota^{(DL,L)}_{\triangle_{e_1}} \Psi \right). \tag{4.113}$$

Applying the anchor on the last equality

$$[X_{e_1}, X_{e_2}] = X_{\{e_1,e_2\}} + \sigma \circ \hat{J}^\sharp \left(\iota^{(DL,L)}_{\triangle_{e_2}} \iota^{(DL,L)}_{\triangle_{e_1}} \Psi \right), \tag{4.114}$$

i.e., the characteristic distribution is involutive. □

Theorem 4.4.3. *The characteristic distribution corresponding to a Jacobi bundle with background $(L \to M, J, \Psi)$, $\mathcal{K}_J$, is completely Stefan-Sussmann [71, 72] integrable.*

Proof. We can prove the theorem in two fashions. First, passing to local picture, the Jacobi bundle with background reduces to a Jacobi pair with background whose characteristic distribution has been proved to be integrable (see Theorem 3.4.7). Second, by using Proposition 4.4.2, in the light of the holonomy groupoid associated with the characteristic distribution [2], the integrability emerges. □

Transitive Jacobi bundles with background. Here, we shall analyze the transitive Jacobi bundles. In the light of Remarks 4.2.5 and 4.2.8 it results that the analysis essentially depends on the dimension parity of the base manifold. Invoking a similar analysis in the previous chapter and sections of the present chapter, we expect to exhibit kinds of locally conformal symplectic/ contact structures that encode transitive Jacobi bundles with background.

Definition 4.4.4. *A* locally conformal symplectic structure with background *on a given line bundle $L \to M$ is a pair $\left((\nabla, \omega), \left(\hat{\psi}, \hat{\omega}\right)\right)$ consisting of a representation ∇ of the tangent Lie algebroid $(TM \to M, [\bullet, \bullet], \mathrm{id})$ on a line bundle, two L-valued 2-forms $\omega, \hat{\omega} \in \Omega^2(M; L)$ among which ω is non-degenerate and an L-valued 3-form $\hat{\psi} \in \Omega^3(M; L)$ that verify the consistency condition*

$$d_\nabla (\omega - \hat{\omega}) = \hat{\psi}. \tag{4.115}$$

In the previous definition, we denoted by d_∇ the homological degree 1 derivation associated with the Jacobi algebroid structure $([\bullet, \bullet], \nabla)$ on the pair (TM, L) (see the third statement in Theorem 4.1.4).

Remark 4.4.5. *The non-degeneracy of the 2-form ω implies that the base manifold is necessary even-dimensional.*

It is worth noticing that when the line bundle is trivial (4.17) then the locally conformal symplectic structure with background reduces to a locally conformal symplectic pair with background [16, 19, 20]. The manifold endowed with such a pair being nothing but a locally conformal symplectic manifold with background [16, 19, 20]. Indeed, when the line bundle is trivial (4.17), accordingly with Remark 4.1.5 the flat connection ∇ is given by the closed 1-form $\alpha \in \Omega^1(M)$, $\alpha := \omega_\nabla$ via (4.19). Moreover, in the same context, the module $\Omega^\bullet(M; L)$ over the graded commutative algebra $\Omega^\bullet(M)$ reduces to $\Omega^\bullet(M)$ over the same graded commutative algebra, which further implies that the L-valued forms in Definition 4.4.4 are exterior forms on the smooth manifold, $\omega, \hat{\omega} \in \Omega^2(M)$ and $\hat{\psi} \in \Omega^3(M)$. Within this framework, the homological degree 1 derivation (4.25), which will be denoted by d_α, reduces to (4.97). Putting together the previous information, the considered locally conformal symplectic structure with background consists of the geometric objects

$$\alpha \in \Omega^1(M), \quad \omega, \hat{\omega} \in \Omega^2(M), \quad \hat{\psi} \in \Omega^3(M),$$

among which α is closed and ω is non-degenerate. In addition, the previous objects verify the consistency condition

$$\mathrm{d}(\omega - \hat{\omega}) + \alpha \wedge (\omega - \hat{\omega}) = \hat{\psi}.$$

This means that $\left((\omega, \alpha), \left(\hat{\psi} + \mathrm{d}\hat{\omega}, \hat{\omega}\right)\right)$ is nothing but a locally conformal symplectic pair with background [16, 19, 20].

Proposition 4.4.6. *Let $\left((\nabla, \omega), \left(\hat{\psi}, \hat{\omega}\right)\right)$ a locally conformal symplectic structure with background on the line bundle $L \to M$. Then, there exists a canonically associated transitive Jacobi bundle with background $(L \to M, J, \Psi)$.*

Proof. Starting with the non-degenerate L-valued 2-form ω, we introduce the Hamiltonian vector fields by (4.65) and the first-order bi-differential operator J via (4.67). Also, by direct computation we get

$$J(e_1, e_2) = \langle \mathrm{d}_\nabla e_2, X_{e_1} \rangle = \nabla_{X_{e_1}} e_2, \tag{4.116}$$

which shows that

$$\triangle_e := \hat{J}^\sharp(j^1 e) = \nabla_{X_e}, \quad \left(\sigma \circ \hat{J}^\sharp\right)(j^1 e) = X_e. \tag{4.117}$$

Invoking the vector bundle isomorphism (4.64) again, the last result further proves that the bi-differential operator (4.116) associated with the considered locally conformal symplectic structure with background enjoys (4.95). Moreover, the bi-symbol associated with the bi-differential operator (4.67) coincides with the inverse of (4.64), i.e.,

$$\tilde{J}^\sharp = \omega^\sharp, \tag{4.118}$$

that further leads to

$$TM = \operatorname{Im} \omega^\sharp = \operatorname{Im} \tilde{J}^\sharp \subseteq \operatorname{Im}(\sigma \circ \hat{J}^\sharp) \subseteq TM \tag{4.119}$$

which eventually means that the bi-differential operator associated to the considered locally conformal symplectic structure with background verifies (4.61).

At this stage, using the pull-back of the symbol map σ, from the remaining L-valued forms involved in the considered structure (see Definition 4.4.4), we introduce the L-valued forms in Atiyah der-complex

$$\tilde{\omega} := \left(\wedge^2 \sigma\right)^* \hat{\omega} \in \Omega_L^2, \quad \tilde{\psi} := \left(\wedge^3 \sigma\right)^* \hat{\psi} \in \Omega_L^3, \tag{4.120}$$

that allow the construction of

$$\Psi := \mathrm{d}_L \tilde{\omega} + \tilde{\psi}. \tag{4.121}$$

With all this preparation at hand, we are ready to show that the consistency condition (4.115) implies that the bi-differential operator (4.116) enjoys of (4.103), i.e., it is a Jacobi structure with the background Ψ on the considered line bundle $L \to M$. Indeed, by direct computation we get

$$\begin{aligned}
\langle \mathrm{d}_\nabla \omega, X_{e_1} \wedge X_{e_2} \wedge X_{e_3} \rangle &= \sum_{\text{cyclic}} \left(\nabla_{X_{e_1}} \langle \omega, X_{e_2} \wedge X_{e_3} \rangle - \langle \omega, [X_{e_1}, X_{e_2}] \wedge X_{e_3} \rangle \right) \\
&= \nabla_{X_{e_2}} \nabla_{X_{e_1}} e_3 + \nabla_{X_{e_3}} \nabla_{X_{e_2}} e_1 + \nabla_{X_{e_1}} \nabla_{X_{e_3}} e_2 \\
&= J\left(e_2, J(e_1, e_3)\right) + J\left(e_3, J(e_2, e_1)\right) + J\left(e_1, J(e_3, e_2)\right) \\
&= \frac{1}{2} [\![J, J]\!] \left(e_1, e_2, e_3\right) = \langle \frac{1}{2} [\![J, J]\!], j^1 e_1 \wedge j^1 e_2 \wedge j^1 e_3 \rangle,
\end{aligned}$$

$$\begin{aligned}
\langle \mathrm{d}_\nabla \hat{\omega}, X_{e_1} \wedge X_{e_2} \wedge X_{e_3} \rangle &= \langle \mathrm{d}_\nabla \hat{\omega}, \left(\sigma \circ \hat{J}^\sharp\right)(j^1 e_1) \wedge \left(\sigma \circ \hat{J}^\sharp\right)(j^1 e_2) \wedge \left(\sigma \circ \hat{J}^\sharp\right)(j^1 e_3) \rangle \\
&= \langle \left(\wedge^3 \sigma\right)^* \mathrm{d}_\nabla \hat{\omega}, \triangle_{e_1} \wedge \triangle_{e_2} \wedge \triangle_{e_3} \rangle \\
&= \langle \mathrm{d}_L \tilde{\omega}, \triangle_{e_1} \wedge \triangle_{e_2} \wedge \triangle_{e_3} \rangle \\
&= \langle \left(\wedge^3 \hat{J}^\sharp\right)^* \mathrm{d}_L \tilde{\omega}, j^1 e_1 \wedge j^1 e_2 \wedge j^1 e_3 \rangle,
\end{aligned}$$

and also

$$\begin{aligned}
\langle \hat{\psi}, X_{e_1} \wedge X_{e_2} \wedge X_{e_3} \rangle &= \langle \hat{\psi}, \left(\sigma \circ \hat{J}^\sharp\right)(j^1 e_1) \wedge \left(\sigma \circ \hat{J}^\sharp\right)(j^1 e_2) \wedge \left(\sigma \circ \hat{J}^\sharp\right)(j^1 e_3) \rangle \\
&= \langle \left(\wedge^3 \sigma\right)^* \hat{\psi}, \triangle_{e_1} \wedge \triangle_{e_2} \wedge \triangle_{e_3} \rangle \\
&= \langle \tilde{\psi}, \triangle_{e_1} \wedge \triangle_{e_2} \wedge \triangle_{e_3} \rangle \\
&= \langle \left(\wedge^3 \hat{J}^\sharp\right)^* \tilde{\psi}, j^1 e_1 \wedge j^1 e_2 \wedge j^1 e_3 \rangle.
\end{aligned}$$

Inserting the last three equalities into the consistency condition (4.115), we conclude that the bi-differential operator (4.116) enjoys of (4.103), i.e., $(L \to M, J, \Psi)$ is a Jacobi bundle with background, which, in the light of (4.119) is transitive. □

Remark 4.4.7. *The previous result allows us to conclude that, given a line bundle $L \to M$, every locally conformal symplectic structure with background $((\nabla, \omega), (\hat{\psi}, \hat{\omega}))$ is fully encoded into its associated transitive Jacobi structure with background (J, Ψ). Indeed, from* (4.116) *we can read both the TM-connection on the line bundle and the Hamiltonian vector fields X_e corresponding to sections in the line bundle, $e \in \Gamma(L)$. Also, from* (4.118) *we can read the non-degenerate L-valued 2-form $\omega \in \Omega(M; L)$. Finally, in the light of* (4.42) *supplemented with the injectivity of σ^*, from* (4.120) *we can extract the remaining L-valued forms $\hat{\omega} \in \Omega^2(M; L)$ and $\hat{\psi} \in \Omega^3(M; L)$, once we have $\Psi \in \Omega^3_L$.*

There exists also the converse of the previous correspondence.

Proposition 4.4.8. *Let $(L \to M, J, \Phi)$ be a transitive Jacobi bundle with background over the even-dimensional base manifold M. There exists a locally conformal symplectic structure with background $\left((\nabla, \omega), \left(\hat{\psi}, \hat{\omega}\right)\right)$ on the same line bundle $L \to M$ that generates (via Proposition 4.4.6) the starting Jacobi structure with background.*

Proof. Initially, we prove that the bi-symbol (4.68) associated with the bi-differential operator J is non-degenerate, i.e., map (4.69) is a vector bundle isomorphism. The skew-symmetry of the bi-symbol (4.68) further implies that the rank of $\tilde{J}^\sharp$ is even-dimensional. On the other hand, the transitivity of the considered structure (4.95) supplemented with the fact that the fibers of DL are one-dimension greater than those of $\operatorname{Im} \hat{J}^\sharp$ exhibits the point-wise decomposition

$$DL = \operatorname{Im} \hat{J}^\sharp \oplus \langle \mathbb{1} \rangle \tag{4.122}$$

and also

$$J^1 L = \operatorname{Ker} \hat{J}^\sharp \oplus \operatorname{Im} \gamma. \tag{4.123}$$

By means of the result (4.70), the last point-wise decomposition of the fibers of the vector bundle J^1L allows to conclude that $\tilde{J}^\sharp$ is a surjective vector bundle morphism, and, moreover a vector bundle isomorphism. Based on this isomorphism, we construct the non-degenerate L-valued 2-form ω, $\omega \in \Omega^2(M; L)$ via

$$\langle \omega, X \wedge Y \rangle := \langle \tilde{J}, \tilde{J}^\flat X \wedge \tilde{J}^\flat Y \rangle, \quad X, Y \in \mathfrak{X}^1(M). \tag{4.124}$$

Denoting by $\tilde{J}^\flat$ the inverse of $\tilde{J}^\sharp$, we define the vector bundle morphism

$$\nabla : TM \to DL, \quad \nabla := \hat{J}^\sharp \circ \gamma \circ \tilde{J}^\flat, \quad \mathfrak{X}^1(M) \ni X \mapsto \nabla_X \in \mathcal{D}^1 L, \tag{4.125}$$

i.e., a TM-connection on the line bundle. We prove that this is a flat one. In view of this, using the definition of Hamiltonian vector fields (4.106) and decomposition (4.123) we get

$$\nabla_{X_e} = \hat{J}^\sharp \circ \gamma \circ \tilde{J}^\flat \circ \sigma \circ \hat{J}^\sharp (j^1 e) = \hat{J}^\sharp (j^1 e) = \triangle_e. \tag{4.126}$$

With the previous result at hand, we prove that (4.125) is a flat connection. By direct computation based on (4.126), (4.113) and (4.114), we derive the curvature associated with the connection (4.125)

$$R_\nabla \left(X_{e_1}, X_{e_2}\right) e_3 = \left(\left[\nabla_{X_{e_1}}, \nabla_{X_{e_2}}\right] - \nabla_{\left[X_{e_1}, X_{e_2}\right]}\right) e_3$$

$$
\begin{aligned}
&= \left([\triangle_{e_1}, \triangle_{e_2}] - \triangle_{\{e_1,e_2\}}\right) e_3 - \langle \Psi, \triangle_{e_1} \wedge \triangle_{e_2} \wedge \triangle_{e_3} \rangle \\
&= \langle \frac{1}{2} [\![J, J]\!], j^1 e_1 \wedge j^1 e_2 \wedge j^1 e_3 \rangle - \langle \Psi, \triangle_{e_1} \wedge \triangle_{e_2} \wedge \triangle_{e_3} \rangle \\
&= \langle \frac{1}{2} [\![J, J]\!] - \left(\wedge^3 \hat{J}^\sharp\right)^* \Psi, j^1 e_1 \wedge j^1 e_2 \wedge j^1 e_3 \rangle = 0,
\end{aligned}
$$

i.e., (4.125) is a flat connection.

It remains to identify the L-valued forms $\hat{\omega}$ and $\hat{\phi}$ from Definition 4.4.4 and to show that they are enjoying the consistency condition (4.115). Due to the acyclicity of Atiyah der-complex (see Remark 4.1.6) it results the decomposition

$$
\Omega_L^3 = \operatorname{Ker}^3 \mathrm{d}_L \oplus \operatorname{Ker}^3 \iota_{\mathbb{1}}^{(DL,L)}
$$

that further gives

$$
\Psi = \mathrm{d}_L \tilde{\omega} + \tilde{\psi}, \tag{4.127}
$$

with

$$
\tilde{\omega} := \iota_{\mathbb{1}}^{(DL,L)} \Psi, \quad \tilde{\psi} := \iota_{\mathbb{1}}^{(DL,L)} \mathrm{d}_L \Psi. \tag{4.128}
$$

Now, due to the injectivity of map σ^*, supplemented with decomposition (4.122) it results that there exists the unique L-valued forms $\hat{\omega} \in \Omega^2(M; L)$ and $\hat{\psi} \in \Omega^3(M; L)$ such that

$$
\tilde{\omega} := \left(\wedge^2 \sigma\right)^* \hat{\omega} \in \Omega_L^2, \quad \tilde{\psi} := \left(\wedge^3 \sigma\right)^* \hat{\psi} \in \Omega_L^3. \tag{4.129}
$$

Finally, we show that the L-valued forms (4.124) and (4.130) verify the consistency condition (4.115). By direct computation, we successively obtain

$$
\begin{aligned}
\langle \mathrm{d}_\nabla \omega, X_{e_1} \wedge X_{e_2} \wedge X_{e_3} \rangle &= \sum_{\text{cyclic}} \left(\nabla_{X_{e_1}} \langle \omega, X_{e_2} \wedge X_{e_3} \rangle - \langle \omega, [X_{e_1}, X_{e_2}] \wedge X_{e_3} \rangle \right) \\
&= \nabla_{X_{e_2}} \nabla_{X_{e_1}} e_3 + \nabla_{X_{e_3}} \nabla_{X_{e_2}} e_1 + \nabla_{X_{e_1}} \nabla_{X_{e_3}} e_2 \\
&= J\left(e_2, J(e_1, e_3)\right) + J\left(e_3, J(e_2, e_1)\right) + J\left(e_1, J(e_3, e_2)\right) \\
&= \frac{1}{2} [\![J, J]\!] \left(e_1, e_2, e_3\right) = \langle \frac{1}{2} [\![J, J]\!], j^1 e_1 \wedge j^1 e_2 \wedge j^1 e_3 \rangle,
\end{aligned}
$$

$$
\begin{aligned}
\langle \mathrm{d}_\nabla \hat{\omega}, X_{e_1} \wedge X_{e_2} \wedge X_{e_3} \rangle &= \langle \mathrm{d}_\nabla \hat{\omega}, \left(\sigma \circ \hat{J}^\sharp\right)(j^1 e_1) \wedge \left(\sigma \circ \hat{J}^\sharp\right)(j^1 e_2) \wedge \left(\sigma \circ \hat{J}^\sharp\right)(j^1 e_3) \rangle \\
&= \langle \left(\wedge^3 \sigma\right)^* \mathrm{d}_\nabla \hat{\omega}, \triangle_{e_1} \wedge \triangle_{e_2} \wedge \triangle_{e_3} \rangle \\
&= \langle \mathrm{d}_L \tilde{\omega}, \triangle_{e_1} \wedge \triangle_{e_2} \wedge \triangle_{e_3} \rangle \\
&= \langle \left(\wedge^3 \hat{J}^\sharp\right)^* \mathrm{d}_L \tilde{\omega}, j^1 e_1 \wedge j^1 e_2 \wedge j^1 e_3 \rangle,
\end{aligned}
$$

and also

$$
\begin{aligned}
\langle \hat{\psi}, X_{e_1} \wedge X_{e_2} \wedge X_{e_3} \rangle &= \langle \hat{\psi}, \left(\sigma \circ \hat{J}^\sharp\right)(j^1 e_1) \wedge \left(\sigma \circ \hat{J}^\sharp\right)(j^1 e_2) \wedge \left(\sigma \circ \hat{J}^\sharp\right)(j^1 e_3) \rangle \\
&= \langle \left(\wedge^3 \sigma\right)^* \hat{\psi}, \triangle_{e_1} \wedge \triangle_{e_2} \wedge \triangle_{e_3} \rangle \\
&= \langle \tilde{\psi}, \triangle_{e_1} \wedge \triangle_{e_2} \wedge \triangle_{e_3} \rangle \\
&= \langle \left(\wedge^3 \hat{J}^\sharp\right)^* \tilde{\psi}, j^1 e_1 \wedge j^1 e_2 \wedge j^1 e_3 \rangle.
\end{aligned}
$$

that prove the consistency condition (4.115). □

The results offered by the last two propositions are compactly written in the theorem below.

Theorem 4.4.9. *Let $L \to M$ be a line bundle and (J, Ψ) be a Jacobi structure with background on it. Then the following conditions are equivalent:*

1. *(J, Ψ) is associated with a (unique) locally conformal symplectic structure with background;*
2. *M is even-dimensional, and the structure (J, Ψ) is transitive;*
3. *the bi-symbol $\tilde{J}$ is non-degenerate.*

In the last part of this section, we analyze transitive Jacobi bundles with background over odd-dimensional base manifolds.

Definition 4.4.10. *Let $L \to M$ be a line bundle and J be a bi-differential operator, $J \in \mathcal{D}^2 L$. The operator is said to be* non-degenerate *if its corresponding 'hat' linear map* (4.58) *is non-degenerate.*

At this point, it is clear that the 'musical' maps associated with the non-degenerate linear map $\hat{J}$ give a one-to-one correspondence between non-degenerate bi-differential operators and non-degenerate Atiyah 2-forms

$$\begin{gathered} \mathcal{D}^2 L \ni J, J - \text{non-degenerate} \leftrightarrow \bar{\Omega} \in \Omega^2_L, \Omega - \text{non-degenerate} \\ \hat{J}^\flat := (\hat{J}^\sharp)^{-1} = \bar{\Omega}^\flat, \end{gathered} \tag{4.130}$$

which further leads to

$$\mathrm{d}_L \bar{\Omega} = \frac{1}{2} (\wedge^3 \hat{J}^\flat)^* [\![J, J]\!]. \tag{4.131}$$

The last relation can be derived by direct computation, as follows.

$$\begin{aligned} \langle \mathrm{d}_L \bar{\Omega}, \triangle_{e_1} \wedge \triangle_{e_2} \wedge \triangle_{e_3} \rangle &= \sum_{\text{cyclic}} (\triangle_{e_1} \langle \bar{\Omega}, \triangle_{e_2} \wedge \triangle_{e_3} \rangle - \langle \bar{\Omega}, [\triangle_{e_1}, \triangle_{e_2}] \wedge \triangle_{e_3} \rangle) \\ &= \sum_{\text{cyclic}} (\triangle_{e_1} \langle \bar{\Omega}^\flat \triangle_{e_3}, \triangle_{e_2} \rangle - \langle \bar{\Omega}^\flat \triangle_{e_3}, [\triangle_{e_1}, \triangle_{e_2}] \rangle) \\ &= - \sum_{\text{cyclic}} \triangle_{e_1} \triangle_{e_2} e_3 = - \frac{1}{2} [\![J, J]\!] (e_1, e_2, e_3) \\ &= - \frac{1}{2} \langle j^1 e_1 \wedge j^1 e_2 \wedge j^1 e_3, [\![J, J]\!] \rangle \\ &= \frac{1}{2} \langle (\wedge^3 \hat{J}^\flat)^* [\![J, J]\!], \triangle_{e_1} \wedge \triangle_{e_2} \wedge \triangle_{e_3} \rangle. \end{aligned}$$

This analysis is systematized in the proposition below.

Proposition 4.4.11. *Let $L \to M$ be a line bundle.*

1. *For any Jacobi structure with background (J, Ψ), the following conditions are equivalent:*

- *(J, Ψ) is non-degenerate, and*
- *(J, Ψ) is transitive and M is odd-dimensional.*

2. *A non-degenerate Jacobi structure with background (J, Ψ) is nothing more than a non-degenerate bi-differential operator J on the considered line bundle.*

3. *If a twisted Jacobi structure (J, Ψ) is non-degenerate, then it is also closed, $d_L\Psi = 0$, with precisely $\Psi = d_L\bar{\Omega}$, where $\bar{\Omega}^\flat = (\hat{J}^\sharp)^{-1}$.*

Proof. 1. The assertions do not take into account the Atiyah 3-form, which means that the proof follows as specified in Remark 4.2.8. 2. By considering a non-degenerate bi-differential operator J, the unique Atiyah 3-form that enjoys of (4.103) is

$$\Psi = \frac{1}{2}(\wedge^3 \hat{J}^\flat)^* [\![J, J]\!].$$

3. The proof has been done in the preamble of the present proposition. □

Remark 4.4.12. *The previous proposition shows that for a given line bundle $L \to M$ over an odd-dimensional manifold, transitive Jacobi structures with background (J, Ψ) coincide with transitive twisted Jacobi structures $(J, \bar{\Omega})$, where $\Psi = d_L\bar{\Omega}$.*

Proposition 4.4.13. *For any twisted contact structure $(\mathcal{H}, \psi)$ on the smooth manifold M, there is a canonically associated non-degenerate, and a fortiori transitive Jacobi structure with background on the quotient line bundle $L := TM/\mathcal{H} \to M$.*

Proof. Let θ be the canonical projection associated with the considered hyperplane distribution (4.72). Inspired by the pure contact case (see Section 4.2), we introduce the $\mathbb{R}$-linear map $\varphi : \mathfrak{X}^1(M) \to \Gamma(\mathcal{H}^* \otimes L)$, $X \mapsto \varphi_X$, via

$$\varphi_X(Y) := -\langle \theta, [X, Y] \rangle - \langle \psi, X \wedge Y \rangle, \quad Y \in \Gamma(\mathcal{H}). \tag{4.132}$$

The previous definition is well-pose, and actually φ is a first-order differential operator. Indeed, by direct computation, we get

$$\varphi_{fX}(Y) - f\varphi_X(Y) = (Yf)\langle \theta, X \rangle, \quad f \in \mathcal{F}(M), \quad X \in \mathfrak{X}^1(M), \quad Y \in \Gamma(\mathcal{H}),$$

or, equivalently

$$\varphi_{fX} - f\varphi_X = \mathrm{d}f|_{\mathcal{H}} \otimes \theta. \tag{4.133}$$

We consider the vector subspace $\mathcal{R}_\mathcal{H}$ of the $\mathbb{R}$-vector space $\mathfrak{X}^1(M)$ defined by

$$\mathcal{R}_\mathcal{H} := \operatorname{Ker} \varphi_\bullet. \tag{4.134}$$

We mention here that $\mathcal{R}_\mathcal{H}$ is not a $\mathcal{F}(M)$-submodule of $\mathfrak{X}^1(M)$ (just as the $\mathbb{R}$-vector subspace of Reeb vector fields (4.75)). With this preparation at hand, the short exact sequence of RR-vector spaces

$$\mathcal{R}_\mathcal{H} \overset{\subseteq}{\rightarrowtail} \mathfrak{X}^1(M) \overset{\varphi}{\twoheadrightarrow} \Gamma(\mathcal{H}^* \otimes L) \tag{4.135}$$

emerges. Due to the fact that

$$\varphi_X = \Omega^\flat X, \quad X \in \Gamma(\mathcal{H})$$

it results that the exact sequence (4.135) is split from the right by $\Omega^\sharp$, i.e.,

$$\varphi \circ \Omega^\sharp = \mathrm{id}_{\Gamma(\mathcal{H}^* \otimes L)} . \tag{4.136}$$

This split exhibits the canonical isomorphism of $\mathbb{R}$-vector spaces

$$\mathfrak{X}^1(M) \xrightarrow{\simeq} \mathcal{R}_\mathcal{H} \oplus \Gamma(\mathcal{H}), \quad X \overset{\simeq}{\mapsto} (X - \Omega^\sharp\varphi_X) + \Omega^\sharp\varphi_X. \tag{4.137}$$

Using the projection on the first term in the decomposition (4.137), the linear map $\theta \in \Omega(M; L)$, which particularly is an $\mathbb{R}$-linear map of $\mathbb{R}$-vector spaces, and invoking the universality of quotient space, we get the $\mathbb{R}$-linear map that closes the diagram

$$\begin{array}{ccc} \mathfrak{X}^1(M) & \xrightarrow{\theta} & \Gamma(L) \\ {\scriptstyle p_1}\downarrow & \swarrow {\scriptstyle \mathcal{X}_\bullet} & \\ \mathcal{R}_\mathcal{H} & & \end{array} \tag{4.138}$$

The map

$$\mathcal{X}_\bullet : \Gamma(L) \longrightarrow \mathcal{R}_\mathcal{H} \subset \mathfrak{X}^1(M), \quad e \mapsto \mathcal{X}_e \tag{4.139}$$

is uniquely determined by

$$\langle \theta, \mathcal{X}_e \rangle = e. \tag{4.140}$$

This is not linear (with respect to the commutative algebra $\mathcal{F}(M)$), but it is a first-order differential operator between the vector bundles $L \to M$ and $TM \to M$

$$\mathcal{X}_{fe} - f\mathcal{X}_e = -\Omega^\sharp(\mathrm{d}f|_\mathcal{H} \otimes e), \quad f \in \mathcal{F}(M), \quad e \in \Gamma(L). \tag{4.141}$$

As a by-product, the previous result ensures also that

$$\langle \{(\mathcal{X}_e)_x : e \in \Gamma(L)\} \rangle = T_xM, \quad x \in M. \tag{4.142}$$

With all these preparations at hand, we are in position to introduce the $\mathbb{R}$-biliniar and skew-symmetric bracket

$$\{e_1, e_2\} := \langle \theta, [\mathcal{X}_{e_1}, \mathcal{X}_{e_2}] \rangle + \langle \psi, \mathcal{X}_{e_1} \wedge \mathcal{X}_{e_2} \rangle, \quad e_1, e_2 \in \Gamma(L). \tag{4.143}$$

By direct computation based on (4.137), (4.140) and (4.141), we can show that (4.143) is a bi-differential operator

$$\begin{aligned} \{e_1, fe_2\} &= \langle \theta, [\mathcal{X}_{e_1}, \mathcal{X}_{fe_2}] \rangle + \langle \psi, \mathcal{X}_{e_1} \wedge \mathcal{X}_{fe_2} \rangle \\ &= \langle \theta, \left[\mathcal{X}_{e_1}, f\mathcal{X}_{e_2} - \Omega^\sharp(\mathrm{d}f|_\mathcal{H} \otimes e_2)\right] \rangle + \langle \psi, X_{e_1} \wedge (fX_{e_2} - \Omega^\sharp(\mathrm{d}f|_\mathcal{H} \otimes e_2)) \rangle \\ &= \langle \theta, [\mathcal{X}_{e_1}, f\mathcal{X}_{e_2}] \rangle + \langle \psi, \mathcal{X}_{e_1} \wedge f\mathcal{X}_{e_2} \rangle + \varphi_{X_{e_1}}(\Omega^\sharp(\mathrm{d}f|_\mathcal{H} \otimes e_2)) \\ &= f\{e_1, e_2\} + (\mathcal{X}_{e_1} f)e_2. \end{aligned}$$

If we denote with $J := \{\bullet, \bullet\}$, the previous reasoning shows that $J \in \mathcal{D}^2 L$. In addition, due to

$$(\sigma \circ \hat{J}^\sharp)(j^1 e) = \mathcal{X}_e, \quad e \in \Gamma(L),$$

in the light of (4.142), we conclude that J is non-degenerate. Invoking now Proposition 4.4.11, the proof is completed. □

Proposition 4.4.14. *Let (J, Ψ) be a Jacobi structure with background on a line bundle $L \to M$. Then the following conditions are equivalent:*

1. *there exists a twisted contact structure $(\mathcal{H}, \psi)$, and a vector-bundle isomorphism $L \simeq TM/\mathcal{H}$, such that $(J, \Psi = d_L \bar{\Omega})$ is the twisted Jacobi structure canonically associated with $(\mathcal{H}, \psi)$ (according to Proposition 4.4.13),*
2. *M is odd dimensional and the Jacobi structure with background is transitive,*
3. *the first-order bi-differential operator $\hat{J}$ associated with J (4.58) is non-degenerate.*

Proof. The implication 1. $\Rightarrow$ 2. is immediate from Definition 4.3.3, Remark 4.4.11, and Proposition 4.4.13.

The equivalence between statements 2. and 3. preceded Remark 4.4.11.

Now we prove that 2. (or, equivalently 3.) implies 1. Let us denote by $\bar{\Omega} \in \Omega^2_L$ the Atiyah 2-form defined by (4.130). Then, according to Proposition 4.4.11, it results that

$$\Psi = d_L \bar{\Omega}.$$

Further, let us consider the bi-symbol $\tilde{J}$ associated with the considered bi-differential operator J

$$\begin{array}{ccc} J^1 L & \xrightarrow{\hat{J}^\sharp} & DL \\ \sigma^* \big\uparrow & & \big\downarrow \sigma \\ T^*M \otimes L & \xrightarrow{\tilde{J}^\sharp} & TM \end{array}$$

Since $\tilde{J} \in \Gamma(\wedge^2(TM \otimes L^*) \otimes L)$ is skew-symmetric and J is non-degenerate, from the commutative diagram, we can extract the following information:

- the kernel of the linear map $\tilde{J}^\sharp : T^*M \otimes L \to TM$ is the (globally trivial) line subbundle $\operatorname{Ker} \tilde{J}^\sharp \subset T^*M \otimes L$ with a global frame $\theta \in \Omega^1(M; L)$, which is uniquely determined via
$$\sigma^* \theta = -\hat{J}^\flat \mathbb{1}, \tag{4.144}$$
- the image of $\tilde{J}^\sharp : T^*M \otimes L \to TM$, $\mathcal{H} := \operatorname{Im} \tilde{J}^\sharp$, is a hyperplane distribution.

The previous ingredients $\mathcal{H}$ and θ are related via

$$\mathcal{H} = \operatorname{Ker} \theta = \sigma(\langle \mathbb{1} \rangle^{\bar{\Omega}}). \tag{4.145}$$

Indeed, for any $\eta \in \Omega^1(M; L)$, the skew-symmetry of $\tilde{J}$ leads successively to

$$\langle \theta, \tilde{J}^\sharp \eta \rangle = \langle \eta \wedge \theta, \tilde{J} \rangle = -\langle \eta, \tilde{J}^\sharp \theta \rangle = 0.$$

Related to the second equality in (4.145), it comes immediate from Definitions (4.130) and (4.144). If we denote by ω_θ the curvature corresponding to θ, $\omega_\theta \in \Gamma(\wedge^2 \mathcal{H} \otimes L)$, which is uniquely determined by

$$\langle \omega_\theta, X \wedge Y \rangle := \langle \theta, [X, Y] \rangle, \quad X, Y \in \Gamma(\mathcal{H}),$$

then for any two derivations $\Box_{1,2} \in \Gamma(\langle \mathbb{1} \rangle^{\bar{\Omega}}) = \Gamma(\sigma^{-1}(\mathcal{H}))$, by direct computation we get

$$\begin{aligned}\langle \mathrm{d}_L(\sigma^*\theta), \Box_1 \wedge \Box_2 \rangle &= \Box_1 \langle \sigma^*\theta, \Box_2 \rangle - \Box_2 \langle \sigma^*\theta, \Box_1 \rangle - \langle \sigma^*\theta, [\Box_1, \Box_2] \rangle \\ &= \Box_1 \langle \theta, \sigma \Box_2 \rangle - \Box_2 \langle \theta, \sigma_{\Box_1} \rangle - \langle \theta, [\sigma_{\Box_1}, \sigma_{\Box_2}] \rangle \\ &= -\langle \omega_\theta, \sigma_{\Box_1} \wedge \sigma_{\Box_2} \rangle. \end{aligned} \tag{4.146}$$

Further, if we denote by ψ, the L-valued 2-form, $\psi \in \Omega^2(M; L)$, which is uniquely (in the light of the injectivity of σ^*) defined by

$$(\wedge^2 \sigma)^* \psi = -\iota_{\mathbb{1}} \Psi = -\iota_{\mathbb{1}} \mathrm{d}_L \bar{\Omega},$$

then, using the contracting homotopy $\iota_{\mathbb{1}}$ (see second point in Remark 4.1.6), it results

$$\bar{\Omega} = \mathrm{d}_L \iota_{\mathbb{1}} \bar{\Omega} + \iota_{\mathbb{1}} \mathrm{d}_L \bar{\Omega} = \mathrm{d}_L(\sigma^*\theta) - (\wedge^2\sigma)^*\psi \tag{4.147}$$

Now, putting together the results (4.146), (4.147), and the fact that symbol map $\sigma : DL \to TM$ induces a vector bundle isomorphism

$$\frac{\langle \mathbb{1} \rangle^{\bar{\Omega}}}{\langle \mathbb{1} \rangle} \xrightarrow{\simeq} \mathcal{H}$$

we conclude that $\Omega := \omega_\theta + \psi|_{\mathcal{H}} \in \Gamma(\wedge^2 \mathcal{H} \otimes L)$ is non-degenerate, i.e., $(\mathcal{H}, \psi)$ is a *twisted contact structure*.

It remains to show that the starting non-degenerate Jacobi structure with background (J, Ψ) and the Jacobi structure with background associated with $(\mathcal{H}, \psi)$ (see Proposition 4.4.13) coincides.

Notice that, in the current situation, for any section in the line bundle $e \in \Gamma(L)$, there exist two, apparently different, associated Hamiltonian vector fields:

- the Hamiltonian vector field X_e due to the structure (J, Ψ) (see (4.63)), and
- the Hamiltonian vector field $\mathcal{X}_e$ generated by the structure $(\mathcal{H}, \psi)$ (see (4.139)).

As a preliminary step, we initially prove that the two Hamiltonian vector fields are, in fact, identical. First, we show that for any $e \in \Gamma(L)$, X_e is a Reeb vector field, i.e., it belongs to the vector subspace (4.134). Indeed, let Y be a section in the hyperplane distribution, $Y \in \Gamma(\mathcal{H})$. It results that there exists $\Box \in \langle \mathbb{1} \rangle^{\bar{\Omega}}$ such that $Y = \sigma_\Box$. By direct computation, we successively infer

$$\begin{aligned}\varphi_{X_e} &= -\langle \theta, [X_e, Y] \rangle - \langle \psi, X_e \wedge Y \rangle \\ &= -\langle \theta, [\sigma_{\triangle_e}, \sigma_\Box] \rangle - \psi, \sigma_{\triangle_e} \wedge \sigma_\Box \rangle \\ &= -\langle \sigma^*\theta, [\triangle_e, \Box] \rangle - \langle (\wedge^2\sigma)^*\psi, \triangle_e \wedge \Box \rangle \\ &= \langle \bar{\Omega}, \mathbb{1} \wedge [\triangle_e, \Box] \rangle + \langle \mathrm{d}_L \bar{\Omega}, \mathbb{1} \wedge \triangle_e \wedge \Box \rangle \\ &= \langle \bar{\Omega}, \triangle_e \wedge \Box \rangle + \Box \langle \bar{\Omega}, \mathbb{1} \wedge \triangle_e \rangle \\ &= -\Box e + \Box e = 0.\end{aligned}$$

Second, we prove that the Hamiltonian vector field enjoys (4.140). Using the same pedestrian procedure, we derive

$$\langle\theta, X_e\rangle = \langle\sigma^*\theta, \triangle_e\rangle = \langle\sigma^*\theta, \hat{J}^\sharp(j^1 e)\rangle = \langle j^1 e, \hat{J}^\sharp\bar{\Omega}^\flat \mathbb{1}\rangle = e, \quad e \in \Gamma(L).$$

The previous argumentation prove that the specified Hamiltonian vector fields coincide, i.e.,

$$X_e = \mathcal{X}_e, \quad e \in \Gamma(L). \tag{4.148}$$

We close the proof by showing that the brackets associated with (J, Ψ), and $(\mathcal{H}, \psi)$, given by (4.109), and (4.143) respectively, coincide. Indeed, starting from (4.109) and using (4.58), (4.130), (4.147), and (4.148), we successively get

$$\begin{aligned}
\{e_1, e_2\} &= \langle j^1 e_1 \wedge j^1 e_2, \hat{J}\rangle = \langle\bar{\Omega}, \triangle_{e_1} \wedge \triangle_{e_2}\rangle = \langle \mathrm{d}_L(\sigma^*\theta) - (\wedge^2\sigma)^*\psi, \triangle_{e_1} \wedge \triangle_{e_2}\rangle \\
&= \triangle_{e_1}\langle\sigma^*\theta, \triangle_{e_2}\rangle - \triangle_{e_2}\langle\sigma^*\theta, \triangle_{e_1}\rangle - \langle\sigma^*\theta, [\triangle_{e_1}, \triangle_{e_2}]\rangle - \langle\psi, X_{e_1} \wedge X_{e_2}\rangle \\
&= \triangle_{e_1}\langle\theta, X_{e_2}\rangle - \triangle_{e_2}\langle\theta, X_{e_1}\rangle - \langle\theta, [X_{e_1}, X_{e_2}]\rangle - \langle\psi, X_{e_1} \wedge X_{e_2}\rangle \\
&= 2\{e_1, e_2\} - \langle\theta, [X_{e_1}, X_{e_2}]\rangle - \langle\psi, X_{e_1} \wedge X_{e_2}\rangle,
\end{aligned}$$

i.e.

$$\{e_1, e_2\} = \langle\theta, [X_{e_1}, X_{e_2}]\rangle + \langle\psi, X_{e_1} \wedge X_{e_2}\rangle$$

which concludes the proof. □

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